JOURNAL OF ALGEBRA 15, 195–224 (1970)

The Lattice of Equational Classes of Idempotent Semigroups*

J. A. GERHARD

Department of Mathematics, The University of Manitoba, Winnipeg, Canada

Communicated by D. Rees

Received April 10, 1969

The concept of an equational class was introduced by Birkhoff [2] in 1935, and has been discussed by several authors (see for example Tarski [22]). Recently there has been much interest in the subject, and especially in the lattice of equational classes of lattices [1], [8], [11], [12], [17]. However, until now the only nontrivial lattice of equational classes to be completely described was the lattice of equational classes of algebras with one unary operation (Jacobs and Schwabauer [10]). The problem of describing the lattice of equational classes of algebras with a single binary operation, i.e., groupoids. is much more difficult. Kalicki [13] has shown that there are uncountably many atoms in the lattice of equational classes of groupoids. The lattice of equational classes of semigroups (associative groupoids) is uncountable (Evans [5]) and has been investigated by Kalicki and Scott [14], who listed its countably many atoms.

There are only two equational classes of commutative idempotent semigroups, but the removal of either commutativity or idempotence gives rise to a non-trivial lattice. Partial results have been obtained for the lattice of equational classes of commutative semigroups [19], [20]. In the case of idempotent semigroups it is relatively easy to show that the lattice of equational classes has three atoms, and in fact the sublattice generated by the atoms has been shown by Tamura [21] to be the eight-element Boolean lattice. Kimura [15] has described all equations on idempotent semigroups which have at most three variables.

In this paper a complete description of the lattice of equational classes of idempotent semigroups is given. We consider the elements of the lattice to be the equational classes themselves and ignore the foundation problem which this entails. In any case this problem can be easily circumvented, for example by taking as elements of the lattice the fully invariant congruences of the free idempotent semigroup on countably many generators.

* This paper is a revision of the author's doctoral thesis, written under the supervision of Professor G. Bruns, and submitted to McMaster University, Hamilton, Canada.

In Section I the solution of the word problem for free idempotent semigroups given by Green and Rees [9] is described. Several invariants are introduced for use in later parts of the paper.

In Section II, the relation $\sim_n$ is introduced and characterized. The special characterization given in Proposition 2.13 is of central importance, since it leads to the definitions of the relations R_n, S_n, $R_n{}^*$, and $S_n{}^*$. These relations are the chief tools used in classifying equations.

In Section III the equations $(f = g)$ in n variables which satisfy $f \sim_n g$ are singled out for special attention. The classes determined by single equations of this type for fixed n are said to form the n-skeleton of the lattice. The n-skeleton is completely described, using R_n, S_n, $R_n{}^*$ and $S_n{}^*$, for $n \geqslant 3$.

In Section IV the description of the lattice is completed. All equational classes determined by a single equation are related to the various n-skeletons, and the n-skeletons ordered with respect to one another. In this way a description of all equational classes determined by a single equation is given. Finally it is shown that every class of idempotent semigroups is determined by a single equation.

I. Invariants and Free Idempotent Semigroups

Throughout the paper, X will be a fixed countable set, and $F(X)$ the free semigroup generated by X. Every semigroup generated by X may be thought of as consisting of all products $x_1x_2 \cdots x_n$ of finite nonempty sequences in X. The free semigroup $F(X)$ is characterized by the property that two products $x_1x_2 \cdots x_n$, $y_1y_2 \cdots y_m$ are equal if, and only if, $n = m$, and $x_i = y_i$, $(i = 1, 2,..., n)$. We refer to the elements of $F(X)$ as *words* or *terms*.

A semigroup equation is a pair (f, g) with $f, g \in F(X)$. The equation (f, g) is said to *hold*, to be *valid*, or to be *satisfied* in a semigroup S if, and only if, for every homomorphism $\varphi : F(X) \to S$, $\varphi(f) = \varphi(g)$. As is usual we will also say in this case that the equation $(f = g)$ holds in S.

We will be concerned here with idempotent semigroups, i.e. semigroups which satisfy the equation $(x = x^2)$, $(x \in X)$. Let $FI(X)$ be the free idempotent semigroup generated by X. Every homomorphism of $F(X)$ into an idempotent semigroup factors through $FI(X)$ so that X is mapped identically. Let $\kappa : F(X) \to FI(X)$ be the homomorphism which maps X identically. An equation $(f = g)$ holds in every idempotent semigroup if and only if $\kappa(f) = \kappa(g)$. If $\kappa(f) = \kappa(g)$, we write $f \sim g$.

For technical reasons we also introduce $F'(X)$, the free monoid generated by X. $F'(X)$ contains $F(X)$ as a subsemigroup and has an additional element, e, which may be thought of as the product of the empty sequence, and which satisfies $ae = ea = a$ for all $a \in F'(X)$. The definition of $\sim$ may be extended to $F'(X)$ by setting $e \sim e$ and asserting that $e \nsim a$ for all $a \neq e$.

As is well-known [4], p.156, the relation $\sim$ on $F(X)$ can be described as follows:

(1.1) $f \sim g$ if, and only if, there exist $h_0, h_1, \ldots, h_n \in F(X)$ and $p_i, q_i, r_i \in F'(X)$, $(i = 0, 1, \ldots, n-1)$, such that $h_0 = f$, $h_n = g$, and for every $i = 0, 1, \ldots, n-1$, either $h_i = p_i q_i r_i$ and $h_{i+1} = p_i q_i^2 r_i$ or $h_i = p_i q_i^2 r_i$ and $h_{i+1} = p_i q_i r_i$.

The *dual* $(S^*, *)$ of a semigroup $(S, \cdot)$ is defined by $S^* = S$ and $a * b = ba$ for all $a, b \in S$. If $f = x_1 x_2 \cdots x_n \in F(X)$, define $f^* = x_1 * x_2 * \cdots * x_n = x_n x_{n-1} \cdots x_1$. Clearly, $f^{**} = f$, and $f \sim g$ if, and only if, $f^* \sim g^*$.

If $f_i \in F(X)$, $(i = 1, \ldots, n)$, let $\prod_{i=1}^n f_i = f_1 f_2 \cdots f_n$ denote the product taken in $(F(X), \cdot)$. Then $(\prod_{i=1}^n f_i)^* = f_n^* f_{n-1}^* \cdots f_1^*$. We will also be interested in the product of the f_i taken in $(F(X), *)$. This term will be denoted by $(\prod_{i=1}^n f_i)_*$. Thus, $(\prod_{i=1}^n f_i)_* = f_1 * f_2 * \cdots * f_n = f_n f_{n-1} \cdots f_1$. If $x_i \in X$, then $(\prod_{i=1}^n x_i)^* = (\prod_{i=1}^n x_i)_*$. Notice that while $f = gh$ implies $f^* = (gh)^*$; in general, f_* and $(gh)_*$ are not equal.

If $f = x_1 x_2 \cdots x_n \in F(X)$ and $x_i \in X$, $(i = 1, \ldots, n)$, let $L(f) = n$, the length of f, and $E(f) = \{x_1, x_2, \ldots, x_n\}$, the set of variables occurring in f. Let $L(e) = 0$ and $E(e) = \varnothing$.

Since $E(pqr) = E(pq^2r)$, it follows from (1.1) that

$$f \sim g \quad \text{implies} \quad E(f) = E(g). \tag{1.2}$$

For $f = x_1 x_2 \cdots x_n$, $(x_i \in X)$, define $\bar{f}(0) = x_j$ where j is determined by the properties

(i) $i < j$ implies $x_i \neq x_j$

(ii) $E(x_1 \cdots x_j) = E(f)$.

Define $\underline{f}(0) = x_1 \cdots x_{j-1}$. If $|E(f)| = 1$, $\underline{f}(0) = e$. By duality define $\bar{f}(1) = \bar{f^*}(0)$, and $\underline{f}(1) = (\underline{f^*}(0))^*$. Define $\underline{e}(i) = e$ and $\bar{e}(i) = e$, $(i = 0, 1)$. It is an immediate consequence of these definitions that

$$E(f) = E(\underline{f}(0)\,\bar{f}(0)) = E(\bar{f}(1)\,\underline{f}(1)). \tag{1.3}$$

If

$$f \neq e, \quad |E(f)| = |E(\underline{f}(0))| + 1 = |E(\underline{f}(1))| + 1. \tag{1.4}$$

A *substitution* is a homomorphism $\varphi : F(X) \to F(X)$. Every substitution may be extended to equations by defining $\varphi(f, g) = (\varphi(f), \varphi(g))$. We will be especially interested in ***primitive substitutions***, that is, in substitutions φ which map X into itself. If $|E(\varphi(f)) \cup E(\varphi(g))| = n$, then φ is said to be a *substitution in* $(f = g)$ *by* n *variables*. If the equation $(f = g)$ holds in a semigroup S, then every equation obtained from $(f = g)$ by substitution also holds in S. In particular, if $f \sim g$ then $\varphi(f) \sim \varphi(g)$.

(1.5) For every primitive substitution, $\varphi(f^*) = (\varphi(f))^*$.

(1.6) For every substitution, if $E(f) = E(g)$, then $E(\varphi(f)) = E(\varphi(g))$.

(1.7) If $E(f) \neq E(g)$, there exists a primitive substitution φ in $(f = g)$ by two variables such that $E(\varphi(f)) \neq E(\varphi(g))$. In particular, it follows from (1.2) that $\varphi(f) \not\sim \varphi(g)$.

Proof. Assume $b \in E(f) - E(g)$. Choose $a \in X$, $a \neq b$, and define φ by $\varphi(b) = b$, $\varphi(x) = a$, $x \neq b$.

THEOREM 1.8. *(Green and Rees* [9]*) If $f, g \in F(X)$, then $f \sim g$ if, and only if,*

(i) $\bar{f}(0) = \bar{g}(0)$ *and* $\bar{f}(1) = \bar{g}(1)$

(ii) $f(0) \sim g(0)$ *and* $f(1) \sim g(1)$.

Proof. Since this result is in the literature we will only sketch a fairly short proof based on a result of T. C. Brown. In [3] he showed that if $E(g) \subseteq E(f)$, then $f \sim fgf$. From this it follows that if $E(h) \subseteq E(f) = E(g)$, then $fhg \sim (fhg)(fg)(fhg) \sim (fhgf)(gfhg) \sim fg$, (a result also noted by McLean [18]). It is then an easy exercise to show that (i) and (ii) imply $f \sim g$. The converse follows from the fact that $\overline{pqr}(0) = \overline{pqqr}(0)$ and $(pqr)(0) \sim (pq^2r)(0)$.

A mapping π with domain $F(X)$ will be called an *invariant* if $f \sim g$ implies $\pi(f) = \pi(g)$. A set M of invariants will be called a *complete set of invariants* if $f \sim g$ if, and only if, $\pi(f) = \pi(g)$ for all $\pi \in M$. Of course, $\{\kappa\}$, where $\kappa : F(X) \to FI(X)$ maps X identically, is a complete set of invariants. Theorem 1.8 states that the mappings $f \to \bar{f}(0)$, $f \to \bar{f}(1)$, $f \to \kappa(f(0))$, $f \to \kappa(f(1))$ also constitute a complete set of invariants.

Let $F(2)$ be the free semigroup generated by the set $\{0, 1\}$. The remarks made with respect to $F(X)$ also hold for $F(2)$. In particular, we can define the length $L(\alpha)$ of $\alpha \in F(2)$. For $f \in F'(X)$, $\alpha \in F(2)$, we define terms $f(\alpha)$ and $\bar{f}(\alpha)$ by induction on $L(\alpha)$. We defined earlier $f(\alpha)$ and $\bar{f}(\alpha)$ for $L(\alpha) = 1$. Let $L(\alpha) \geqslant 2$, say $\alpha = \beta i$, $(i = 0, 1)$, and define $f(\alpha) = (f(\beta))(i)$ and $\bar{f}(\alpha) = \overline{f(\beta)}(i)$. By induction on $L(\alpha) + L(\beta)$ it is easy to see that, for all $\alpha, \beta \in F(2)$,

$$f(\alpha\beta) = (f(\alpha))(\beta). \tag{1.9}$$

$$\bar{f}(\alpha\beta) = \overline{f(\alpha)}(\beta). \tag{1.10}$$

(For convenience we also define $f(e) = f$.)

Let $\alpha = i_1 i_2 \cdots i_k$, $(i_j \in \{0, 1\})$. Define $\alpha' = i_1' i_2' \cdots i_k'$, where $i_j' = 1 - i_j$. Then

$$f(\alpha') = (f^*(\alpha))^*. \tag{1.11}$$

$$\bar{f}(\alpha') = \overline{f^*}(\alpha). \tag{1.12}$$

From Theorem 1.8 it follows that if $\alpha \in F(2)$ and $f \in F'(X)$, then $f(\alpha) \sim f(\alpha 0) \bar{f}(\alpha 0) \bar{f}(\alpha 1) f(\alpha 1)$. In order to simplify the notation, let

$$\hat{f}(\alpha) = \begin{cases} f(\alpha)\, \bar{f}(\alpha), & \text{if } \alpha = \beta 0, \text{ some } \beta \in F'(2) \\ \bar{f}(\alpha)\, f(\alpha), & \text{if } \alpha = \beta 1, \text{ some } \beta \in F'(2). \end{cases}$$

Then $f(\alpha) \sim \hat{f}(\alpha 0) \hat{f}(\alpha 1)$.

$$\hat{f}(\alpha') = ((f^*)^\wedge (\alpha))^*. \tag{1.13}$$

If

$$L(\alpha) \leqslant |E(f)|, \quad \text{then} \quad |E(f)| = |E(f(\alpha))| + L(\alpha). \tag{1.14}$$

If

$$L(\alpha) = |E(f)|,\ \alpha = i_1 \cdots i_k, \quad \text{then} \quad E(f) = \{\bar{f}(i_1), \bar{f}(i_1 i_2), \ldots, \bar{f}(i_1 i_2 \cdots i_k)\}. \tag{1.15}$$

It follows by induction from Theorem 1.8 that

COROLLARY 1.16. *The set $\{f \to \bar{f}(\alpha) \mid \alpha \in F(2)\}$ is a complete set of invariants.*

Since E and $f \to \bar{f}(\alpha)$ (for any $\alpha \in F(2)$) are invariants, it follows in particular that the mappings H and H^*, defined by $H(f) = \bar{f}(0^k)$ and $H^*(f) = H(f^*) = \bar{f}(1^k)$ (where $k = |E(f)|$) are invariants. If $f = x_1 \cdots x_n$, $H(f) = x_1$ and $H^*(f) = x_n$.

The mappings I and F, defined by $I(f) = \bar{f}(0^k) \bar{f}(0^{k-1}) \cdots \bar{f}(0)$ and $F(f) = (I(f^*))^* = \bar{f}(1) \bar{f}(1^2) \cdots \bar{f}(1^k)$, (where $|E(f)| = k$), are likewise invariants. $I(f)$ indicates the order of first occurrance of variables in f.

The mappings $\bar{I}$ and $\bar{F}$, defined by $\bar{I}(f) = \hat{f}(0)$ and $\bar{F}(f) = (\bar{I}(f^*))^* = \hat{f}(1)$, are not themselves invariants, but the composed mappings $\kappa \circ \bar{I}$ and $\kappa \circ \bar{F}$ are invariants.

H, I, E, and F were introduced by Tamura [21].

For notational convenience, we define the following for $n \geqslant 0$:

$$H^n = \begin{cases} H \text{ if } n \text{ is even} \\ H^* \text{ if } n \text{ is odd}, \end{cases} \qquad I^n = \begin{cases} I \text{ if } n \text{ is even} \\ F \text{ if } n \text{ is odd}, \end{cases} \qquad \bar{I}^n = \begin{cases} \bar{I} \text{ if } n \text{ is even} \\ \bar{F} \text{ if } n \text{ is odd}. \end{cases}$$

From these definitions, $(H^n)^* = H^{n+1}$, $(I^n(f^*))^* = I^{n+1}(f)$, and $(\bar{I}^n(f^*))^* = \bar{I}^{n+1}(f)$, for all $n \geqslant 0$.

If φ is any substitution, it is clear that $H^n(\varphi(f)) = H^n(\varphi(H^n(f))$. In particular (for every $n \geqslant 0$),

$$H^n(f) = H^n(g) \quad \text{implies} \quad H^n(\varphi(f)) = H^n(\varphi(g)). \tag{1.17}$$

(1.18) If $H^n(f) \neq H^n(g)$, then there exists a primitive substitution φ in $(f = g)$ by two variables such that $H^n(\varphi(f)) \neq H^n(\varphi(g))$, and in particular such that $\varphi(f) \not\sim \varphi(g)$.

(1.19) Let φ be a primitive substitution, and let k be the largest number with the property that $\varphi(\bar{f}(0^k)) = \overline{\varphi(f)}(0)$. Then $(\varphi(f))^\wedge(0) = \varphi(\hat{f}(0^k))$.

Proof. From the definition of k it follows that $\overline{\varphi(f)}(0) = \varphi(\bar{f}(0^k))$ and $\overline{\varphi(f)}(0) \notin E(\varphi(f(0^k)))$. Therefore, $(\varphi(f))^\wedge(0) = \varphi(\hat{f}(0^k))$.

(1.19)* Let φ be a primitive substitution, and let k be the largest number with the property that $\varphi(\bar{f}(1^k)) = \overline{\varphi(f)}(1)$. Then $(\varphi(f))^\wedge(1) = \varphi(\hat{f}(1^k))$.

(1.20) $I^n(\varphi(f)) = I^n(\varphi(I^n(f)))$, for any primitive substitution φ.

Proof. We will show by induction on $|E(\varphi(f))|$ that $I(\varphi(f)) = I(\varphi(I(f)))$. If $|E(\varphi(f))| = 1$, the result is trivial. In the general case, (1.19) states that $(\varphi(f))^\wedge(0) = \varphi(\hat{f}(0^k))$, for some $k \geqslant 1$. By inductive hypothesis we can assume that $I(\varphi(f(0^k))) = I(\varphi(I(f(0^k))))$. Moreover, it follows from the definitions that $\overline{\varphi(I(f))}(0) = \overline{\varphi(f)}(0) = \varphi(\bar{f}(0^k))$. Since it is obvious that $I(f) = I(\hat{f}(0))$ for any f, we can use the above results and (1.19) to show that $I(\varphi(f)) = I(\varphi(I(f)))$. The proof that $F(\varphi(f)) = F(\varphi(F(f)))$ follows in a similar way from (1.19)*.

Statement (1.20) has as an immediate corollary

(1.21) $I^n(f) = I^n(g)$ implies $I^n(\varphi(f)) = I^n(\varphi(g))$.

(1.22) *If* $I^n(f) \neq I^n(g)$, then there exists a primitive substitution φ in $(f = g)$ by at most three variables such that $I^n(\varphi(f)) \neq I^n(\varphi(g))$ and, hence, in particular such that $\varphi(f) \not\sim \varphi(g)$.

Proof. We will only prove the statement for n even. The statement for n odd follows since $F(f) = (I(f^*))^*$. By (1.7) and (1.18) we can assume that $E(f) = E(g)$ and $H(f) = H(g)$. Since $I(f) \neq I(g)$, there exist $b, c \in X$, $b \neq c$, and $f_i, g_i \in F'(x)$, $(i = 1, 2, 3)$, such that $I(f) = f_1 b f_2 c f_3$, and $I(g) = g_1 c g_2 b g_3$. Let φ be the substitution defined by $\varphi(x) = a$, $(x \neq b, c)$; $\varphi(b) = b$, $\varphi(c) = c$. Then $\varphi(f) = abf'$ and $\varphi(g) = acg'$, some $f', g' \in F(X)$, and therefore $I(\varphi(f)) \neq I(\varphi(g))$.

The equational class of idempotent semigroups which is determined by the equation $(f = g)$ will be denoted by $[f = g]$. Similarly, if $(f_\alpha = g_\alpha)_{\alpha\in I}$ is a family of equations, $[(f_\alpha = g_\alpha)_{\alpha\in I}]$ will denote the equational class of idempotent semigroups determined by the family.

LEMMA 1.23. *If* $E(f) = E(g)$, *then*

$$[f = g] = [(f = \bar{I}(f)\bar{F}(g)), (g = \bar{I}(f)\bar{F}(g))].$$

Proof. It is clear that the right side of the above equation is a subclass of the left. Conversely, if $(f = g)$ holds in an idempotent semigroup S, then in $S, f = \bar{I}(f)f = \bar{I}(f)\bar{I}(g)\bar{F}(g) = \bar{I}(f)\bar{F}(g)$. Similarly, the other equation also holds in S.

The dual $\mathfrak{A}^*$ of a class $\mathfrak{A}$ of semigroups is the class defined by $S \in \mathfrak{A}^*$ if, and only if, $S^* \in \mathfrak{A}$. Then

$$[f = g]^* = [f^* = g^*]. \tag{1.24}$$

$$[f = g] \subseteq [p = q] \quad \text{iff} \quad [f = g]^* \subseteq [p = q]^*. \tag{1.25}$$

II. THE RELATIONS $\sim_n$, R_n, S_n, R_n^*, S_n^*

1. *The Relation* $\sim_n$

DEFINITION 2.1. $f \sim_n g$ iff $\varphi(f) \sim \varphi(g)$ for every primitive substitution φ in $(f = g)$ by less than n variables.

For all n, the relation $\sim_n$ is a congruence relation on $F(X)$. In fact, the transitivity of the relation $\sim_n$ is the only nontrivial part of the statement. Transitivity is a consequence of the fact that (for $n \geqslant 3$) $f \sim_n g$ implies $E(f) = E(g)$, (Proposition 2.4, below).

Since $x \sim x^2$, it follows that $f \sim_2 g$ for all $f, g \in F(X)$. Moreover

$$\begin{aligned} &\text{If } f \sim_n g, \quad \text{then} \quad f \sim_k g \text{ for all } k \leqslant n.\\ &\text{If } f \sim g, \quad \text{then} \quad f \sim_k g \text{ for all } k. \end{aligned} \tag{2.2}$$

$$\text{If } |E(f) \cup E(g)| < n, \quad \text{then} \quad f \sim_n g \text{ iff } f \sim g. \tag{2.3}$$

2. *A Characterization of* $\sim_n$ *for* $n \geqslant 3$

PROPOSITION 2.4. $f \sim_3 g$ *iff* $E(f) = E(g)$, $H(f) = H(g)$ and $H^*(f) = H^*(g)$.

Proof. Let φ be any primitive substitution in $(f = g)$ by two variables. Then $\varphi(f) \sim H(\varphi(f))\,\overline{\varphi(f)}(0)\,\overline{\varphi(f)}(1)\,H^*(\varphi(f))$. If $E(f) = E(g)$, $H(f) =$

$H(g)$, and $H^*(f) = H^*(g)$, the same holds for $\varphi(f)$ and $\varphi(g)$, (by (1.6) and (1.17)). But then $\overline{\varphi(f)}(0) = E(\varphi(f)) - \{H(\varphi(f))\} = \overline{\varphi(g)}(0)$ and, similarly, $\overline{\varphi(f)}(1) = \overline{\varphi(g)}(1)$. Therefore, $\varphi(f) \sim \varphi(g)$ and hence $f \sim_3 g$.

The converse implication follows from (1.7) and (1.18).

PROPOSITION 2.5. *For $n \geqslant 4$, $f \sim_n g$ iff*

(i) *$I(f) = I(g)$ and $F(f) = F(g)$*

(ii) *$f(0^r) \sim_{n-1} g(0^r)$ and $f(1^r) \sim_{n-1} g(1^r)$ for every $r \geqslant 1$.*

Proof. Assume first that (i) and (ii) hold, and let φ be any primitive substitution in $(f = g)$ by less than n variables. From (1.21) it follows that $I(\varphi(f)) = I(\varphi(g))$, and in particular that $\overline{\varphi(f)}(0) = \overline{\varphi(g)}(0)$. Moreover, by (1.19), $(\varphi(f))(0) = \varphi(f(0^k))$, where k is the largest number with the property that $\varphi(\bar{f}(0^k)) = \overline{\varphi(f)}(0)$. Since k depends only on $I(f)$ and $\overline{\varphi(f)}(0)$, it follows that k is also the largest number such that $\varphi(\bar{g}(0^k)) = \overline{\varphi(g)}(0)$, and that $(\varphi(g))(0) = \varphi(g(0^k))$. Now φ is a primitive substitution in $(f(0^k) = g(0^k))$ by less than $n - 1$ variables, and since $f(0^k) \sim_{n-1} g(0^k)$, it follows that $\varphi(f(0^k)) \sim \varphi(g(0^k))$. Therefore by (1.19),

$$I(\varphi(f)) = \varphi(f(0^k))\,\varphi(\bar{f}(0^k)) \sim I(\varphi(g)).$$

By duality (using (1.11) and the definition of F and $\bar{F}$), we can also show that $\bar{F}(\varphi(f)) \sim \bar{F}(\varphi(g))$. Therefore $\varphi(f) \sim \varphi(g)$, and hence $f \sim_n g$.

Conversely, assume $f \sim_n g$. Since $n \geqslant 4$, it follows from (1.22) that $I(f) = I(g)$ and $F(f) = F(g)$. Assume $f(0^r) \not\sim_{n-1} g(0^r)$ for some $r \geqslant 1$. Then there exists a primitive substitution φ in $(f(0^r) = g(0^r))$ by less than $n - 1/\bar{f}(0^r)$ variables such that $\varphi(f(0^r)) \not\sim \varphi(g(0^r))$. We can assume $\varphi(x) = \bar{f}(0^r)$ for all $x \notin E(f(0^r))$. Then φ is a primitive substitution in $(f = g)$ by less than n variables. Moreover $(\varphi(f))(0) = \varphi(f(0^r)) \not\sim \varphi(g(0^r)) = (\varphi(g))(0)$, and therefore $\varphi(f) \not\sim \varphi(g)$, which contradicts $f \sim_n g$. We have, therefore, shown that $f(0^n) \sim_{n-1} g(0^r)$ for all $r \geqslant 1$. That $f(1^r) \sim_{n-1} g(1^r)$ can be shown by duality, completing the proof.

Let $\alpha \in F(2)$. Then $\alpha = i_1^{n_1} i_2^{n_2} \cdots i_k^{n_k}$ where $i_j \neq i_{j+1}$, $(1 \leqslant j < k)$, and $n_j \geqslant 1$, $(1 \leqslant j \leqslant k)$. Define $\bar{\alpha} = i_1 i_2 \cdots i_k$.

COROLLARY 2.6. *Let $n \geqslant 4$, $f \sim_n g$. Then $f(\alpha) \sim_{n-L(\bar{\alpha})} g(\alpha)$.*

3. *A Characterization of $p \sim_n q$ if $E(p) = E(q)$ and $|E(p)| = n \geqslant 3$*

Let $\sigma_n = \prod_{i=1}^n a_i$, where $a_i = 0$ for i odd and $a_i = 1$ for i even, and let $\sigma_0 = e$. Then σ_n is called the standard sequence of length n. According to our earlier definitions, $\sigma_n' = \prod_{i=1}^{n} a_i'$, where $a_i' = 1$ for i odd and $a_i' = 0$ for i even. For any $n \geqslant 0$, we can prove that

$$\begin{aligned} 1\sigma_n &= \sigma'_{n+1} \\ 0\sigma_n' &= \sigma_{n+1}\,. \end{aligned} \tag{2.7}$$

$$|E(p)| = n \geqslant 3 \qquad \text{implies} \quad |E(\hat{p}(\sigma_{n-2}))| = |E(\hat{p}(\sigma'_{n-2}))| = 3. \tag{2.8}$$

$$\begin{aligned} (p(0))^{\wedge}\,(\sigma'_{n-2}) &= \hat{p}(\sigma_{n-1}) \\ (p(1))^{\wedge}\,(\sigma_{n-2}) &= \hat{p}(\sigma'_{n-1}). \end{aligned} \tag{2.9}$$

Proof. Statement (2.9) follows from (2.7), (1.9), (1.10), and the fact that $H^*(\sigma'_{n-2}) = H^*(\sigma_{n-1})$ and $H^*(\sigma_{n-2}) = H^*(\sigma'_{n-1})$.

Let P be a product in $F(X)$. Then P is said to be an *expansion for f* if $f \sim P$. In the expansions for f with which we will be concerned, terms occur which make the following definitions convenient. For any $\alpha \in F'(2)$,

$$f^{\#}(\alpha 1) = \hat{f}(\alpha 0)\,\bar{f}(\alpha 1)$$

$$f^{\#}(\alpha 0) = \bar{f}(\alpha 0)\,\hat{f}(\alpha 1).$$

The term $f^{\#}(\alpha)$ is defined in such a way that it will occur in some expansion for f obtained by successive applications of Theorem 1.8. Thus

$$f(\alpha) = f(\alpha 0) f^{\#}(\alpha 0) = f^{\#}(\alpha 1) f(\alpha 1) \tag{2.10}$$

$$f^{\#}(\alpha') = ((f^*)^{\#}(\alpha))^*. \tag{2.11}$$

We will be most interested in $f^{\#}(\sigma_i)$.

$$f^{\#}(\sigma_i) = \begin{cases} \hat{f}(\sigma_{i-1}0)\,\bar{f}(\sigma_i) & \text{if } i \text{ is even} \\ \bar{f}(\sigma_i)\,\hat{f}(\sigma_{i-1}1) & \text{if } i \text{ is odd.} \end{cases}$$

LEMMA 2.12. *For every $r \geqslant 0$, the following are expansions for f:*

(i) $$\left(\prod_{i=1}^{r} f^{\#}(\sigma_{2i})\right) f(\sigma_{2r+1}) \left(\prod_{j=0}^{r} f^{\#}(\sigma_{2j+1})\right)_{*}.$$

(ii) $$\left(\prod_{i=1}^{r+1} f^{\#}(\sigma_{2i})\right) f(\sigma_{2r+2}) \left(\prod_{j=0}^{r} f^{\#}(\sigma_{2j+1})\right)_{*}.$$

Proof. The proof is by induction on r. If $r = 0$, (i) states that $f \sim f(0) f^{\#}(0)$, which follows from Theorem 1.8. Assume (i) holds for some $r \geqslant 0$. Now $f(\sigma_{2r+1}) \sim f^{\#}(\sigma_{2r+2}) f(\sigma_{2r+2})$ and therefore (ii) holds for this r. Similarly, if (ii) holds for some $r \geqslant 0$ then since $f(\sigma_{2r+2}) \sim f(\sigma_{2r+3}) f^{\#}(\sigma_{2r+3})$, (i) holds for $r + 1$.

We will be interested in expansions for f in which $\hat{f}(\sigma_{n-2})$ occurs. If n is odd, Lemma 2.12 (i) gives such an expansion for $r = \frac{1}{2}(n-3)$, and if n is even, Lemma 2.12 (ii) gives such an expansion if $r = \frac{1}{2}(n-4)$. These two results can be combined to show that

$$A_n{}^0(f) = \left(\prod_{i=1}^{[1/2(n-2)]} f^{\#}(\sigma_{2i})\right) \hat{f}(\sigma_{n-2}) \left(\prod_{j=0}^{[1/2(n-3)]} f^{\#}(\sigma_{2j+1})\right)_*$$

(where $[x]$ is the integral part of x) is an expansion for f in which $\hat{f}(\sigma_{n-2})$ occurs.

An expansion for f in which $\hat{f}(\sigma'_{n-2})$ occurs may be given by duality, by defining $A_n{}^1(f) = (A_n{}^0(f^*))^*$. It is, of course, an easy exercise to give an explicit formula for $A_n{}^1(f)$ using (1.11) and (1.12).

The *standard expansion of order n for f* is $A_n(f) = \bar{I}(A_n{}^0(f))\bar{F}(A_n{}^1(f))$. By Theorem 1.8, $A_n(f)$ is in fact an expansion for f. Moreover, it can be shown that

$$\bar{I}(A_n{}^0(f)) = \left(\prod_{i=1}^{[1/2(n-2)]} f^{\#}(\sigma_{2i})\right) \hat{f}(\sigma_{n-2}) \left(\prod_{j=1}^{[1/2(n-3)]} f^{\#}(\sigma_{2j+1})\right)_* \bar{f}(0).$$

We are now in a position to state and prove the central characterization proposition of Section II.

PROPOSITION 2.13. *Let* $E(p) = E(q)$, $|E(p)| = n \geqslant 3$. *Then* $p \sim_n q$ *iff* (for $\alpha \in F(2)$),

(i) $p(\alpha) \sim q(\alpha)$ *for all* $p(\alpha)$ *occurring in* $A_n(p)$, $q(\alpha)$ *occurring in* $A_n(q)$, *and* $\alpha \neq \sigma_{n-2}$, $\alpha \neq \sigma'_{n-2}$.

(ii) $\bar{p}(\alpha) = \bar{q}(\alpha)$ *for all* $\bar{p}(\alpha)$ *occurring in* $A_n(p)$, $\bar{q}(\alpha)$ *occurring in* $A_n(q)$, *and* $\alpha \neq \sigma_{n-2}$, $\alpha \neq \sigma'_{n-2}$.

(iii) $H^{n+1}(\hat{p}(\sigma_{n-2})) = H^{n+1}(\hat{q}(\sigma_{n-2}))$ *and* $H^n(\hat{p}(\sigma'_{n-2})) = H^n(\hat{q}(\sigma'_{n-2}))$.

Proof. We show first that (i), (ii), and (iii) imply $p \sim_n q$. The proof is by induction on $n \geqslant 3$.

If $n = 3$, condition (iii) states that $H(A_3(p)) = H(A_3(q))$ and $H^*(A_3(p)) = H^*(A_3(q))$. By Proposition 2.4 it follows that $A_3(p) \sim A_3(q)$, and therefore $p \sim q$.

If $n = k + 1 \geqslant 4$, it is sufficient by Proposition 2.5 to show that $I(p) = I(q)$, $F(p) = F(q)$ and, for every $r \geqslant 1$, $p(0^r) \sim_k q(0^r)$ and $p(1^r) \sim_k q(1^r)$.

Since the terms $\bar{p}(0)$, $\bar{p}(00)$, $p(00)$ occur in $A_{k+1}(p)$, it follows by (i) and (ii) that $\bar{p}(0) = \bar{q}(0)$, $\bar{p}(00) = \bar{q}(00)$ and $p(00) \sim q(00)$. It follows from Theorem 1.8 that $p(0^r) \sim q(0^r)$ for all $r \geqslant 2$, and $\bar{p}(0^r) = \bar{q}(0^r)$ for all $r \geqslant 1$, and therefore in particular that $I(p) = I(q)$. It remains to show that $p(0) \sim_k q(0)$.

By inductive hypothesis it is enough to show that (i), (ii), and (iii) hold for $p(0)$ and $q(0)$. If $\alpha = 0\beta$, it follows from $p(00) \sim q(00)$ that $p(0)(\alpha) = p(00)(\beta) \sim q(00)(\beta) = q(0)(\alpha)$, and $\overline{p(0)}(\alpha) = \overline{p(00)}(\beta) = \overline{q(00)}(\beta) = \overline{q(0)}(\alpha)$. Therefore we need only verify conditions (i) and (ii) in case $H(\alpha) = 1$.

If $\overline{p(0)}(\alpha)$ or $p(0)(\alpha)$ occurs in $A_k(p(0))$, if $H(\alpha) = 1$, and if $\alpha \neq \sigma'_{k-2}$, then a straightforward calculation to check the many cases shows that $p(0\alpha)$ and $\bar{p}(0\alpha)$ occur in $A_{k+1}(p)$, that $q(0\alpha)$ and $\bar{q}(0\alpha)$ occur in $A_{k+1}(q)$, and that $0\alpha \neq \sigma_{k-1}$. Therefore $p(0\alpha) \sim q(0\alpha)$ and $\bar{p}(0\alpha) = \bar{q}(0\alpha)$. Hence $p(0)(\alpha) \sim q(0)(\alpha)$ and $\overline{p(0)}(\alpha) = \overline{q(0)}(\alpha)$ for all these α, and therefore (i) and (ii) hold for $p(0)$ and $q(0)$.

By assumption (iii), $H^{k+2}(\hat{p}(\sigma_{k-1})) = H^{k+2}(\hat{q}(\sigma_{k-1}))$. From (2.9), $(p(0))^\wedge(\sigma'_{k-2}) = \hat{p}(\sigma_{k-1})$, and therefore

$$H^k((p(0))^\wedge(\sigma'_{k-2})) = H^{k+2}(\hat{p}(\sigma_{k-1})) = H^k((q(0))^\wedge(\sigma'_{k-2})).$$

Since $p(00) \sim q(00)$, it follows that

$$H^{k+1}((p(0))^\wedge(\sigma_{k-2})) = H^{k+1}((q(0))^\wedge(\sigma_{k-2})).$$

Therefore (iii) holds for $p(0)$ and $q(0)$. By inductive hypothesis therefore we can conclude that $p(0) \sim_k q(0)$.

Using the dual results we can also show that $F(p) = F(q)$ and $p(1^r) \sim_k q(1^r)$ for all $r \geqslant 1$. The proof that (i), (ii), and (iii) imply $p \sim_n q$ is therefore complete.

Conversely, we assume $p \sim_n q$, and show that (i), (ii), and (iii) follow. If $n = 3$, (i) and (ii) are vacuously true, and (iii) follows from Proposition 2.4, since $p \sim \hat{p}(\sigma_1)\, \hat{p}(\sigma_1')$.

Assume $n \geqslant 4$. Let $p(\alpha)$ occur in $A_n(p)$, $\alpha \neq \sigma_{n-2}$, $\alpha \neq \sigma'_{n-2}$. By Corollary 2.6, $p(\alpha) \sim_{n-L(\bar{\alpha})} q(\alpha)$. It can be shown by direct calculation that for these α, $n - L(\bar{\alpha}) \geqslant 3$, and it follows by Proposition 2.4 that $E(p(\alpha)) = E(q(\alpha))$. Moreover for these α, $L(\alpha) > L(\bar{\alpha})$, and therefore $|\, E(p(\alpha)) \cup E(q(\alpha))| < n - L(\bar{\alpha})$. We can conclude (by(2.3)) that $p(\alpha) \sim q(\alpha)$.

If $\bar{p}(\alpha)$ occurs in $A_n(p)$, $\alpha \neq \sigma_{n-2}$, $\alpha \neq \sigma'_{n-2}$, then it can be shown (using Corollary 2.6) that if $\alpha = \beta i^r$ (for some $r \geqslant 1$, $i \in \{0, 1\}$), then $p(\beta) \sim_4 q(\beta)$. By Proposition 2.5, therefore, $\bar{p}(\alpha) = \bar{q}(\alpha)$.

We establish (iii) by induction on $n \geqslant 3$, using (2.9). If $n = 3$, (iii) holds, as we noted above. Assume (iii) for some $k \geqslant 3$. Then $H^{k+2}(\hat{p}(\sigma_{k-1})) = H^k((p(0))^\wedge(\sigma'_{k-2})) = H^k((q(0))^\wedge(\sigma'_{k-2})) = H^{k+2}(\hat{q}(\sigma_{k-1}))$. Similarly, we can show $H^{k+1}(\hat{p}(\sigma'_{k-1})) = H^{k+1}(\hat{q}(\sigma'_{k-1}))$, and therefore by induction (iii) holds.

COROLLARY 2.14. *Let* $|\, E(p)| = n \geqslant 3$, *and* $p \sim_n q$.

(i) $\hat{p}(0) \sim \hat{q}(0)$ *iff* $\hat{p}(\sigma_{n-2}) \sim \hat{q}(\sigma_{n-2})$,

(ii) $\hat{p}(1) \sim \hat{q}(1)$ *iff* $\hat{p}\ \sigma'_{n-2}) \sim \hat{q}(\sigma'_{n-2})$,

and in particular

(iii) $p \sim q$ *iff* $\hat{p}(\sigma_{n-2}) \sim \hat{q}(\sigma_{n-2})$ *and* $\hat{p}(\sigma'_{n-2}) \sim \hat{q}(\sigma'_{n-2})$.

4. *The Relations* R_n, $R_n{}^*$, S_n, $S_n{}^*$

Corollary 2.14 demonstrates the usefulness of comparing $\hat{p}(\sigma_{n-2})$ and $\hat{q}(\sigma_{n-2})$ when p and q satisfy the conditions of the corollary. For p and q, with $p \sim_n q$ but not necessarily satisfying $|E(p)| = n$, we are therefore led to define the following relations.

DEFINITION 2.15. Assume $n \geqslant 3$ and $p \sim_n q$. Then, for every primitive substitution φ in $(p = q)$ by n variables,

(i) pR_nq iff $I^{n+1}((\varphi(p))\hat{}\,(\sigma_{n-2})) = I^{n+1}((\varphi(q))\hat{}\,(\sigma_{n-2}))$,

(ii) $pR_n{}^*q$ iff $p^*R_nq^*$,

(iii) pS_nq iff $(\varphi(p))\hat{}\,(\sigma_{n-2}) \sim (\varphi(q))\hat{}\,(\sigma_{n-2})$,

(iv) $pS_n{}^*q$ iff $p^*S_nq^*$.

LEMMA 2.16. *Assume* $n \geqslant 3$ *and* $p \sim_n q$. *Then for every primitive substitution* φ *in* $(p = q)$ *by* n *variables,*

(i) $p\,R_n{}^*\,q$ *iff* $I^n((\varphi(p))\hat{}\,(\sigma'_{n-2})) = I^n((\varphi(q))\hat{}\,(\sigma'_{n-2}))$,

(ii) $p\,S_n{}^*\,q$ *iff* $(\varphi(p))\hat{}\,(\sigma'_{n-2}) \sim (\varphi(q))\hat{}\,(\sigma'_{n-2})$.

Corollary 2.14 (iii) can now be restated as

(2.17) If $|E(p)| = n \geqslant 3$, then $p \sim q$ iff pS_nq and $pS_n{}^*q$.

Moreover,

$$\begin{aligned} p\,S_n\,q \quad &\text{implies} \quad p\,R_n\,q \qquad \text{for all } n \geqslant 3, \\ p\,S_n{}^*\,q \quad &\text{implies} \quad p\,R_n{}^*\,q \qquad \text{for all } n \geqslant 3. \end{aligned} \tag{2.18}$$

$$p \sim_{n+1} q \text{ iff } pS_nq \text{ and } pS_n{}^*q, \text{ for all } n \geqslant 3. \tag{2.19}$$

Proof. If $p \sim_{n+1} q$, then for every substitution φ in $(p = q)$ by n variables $\varphi(p) \sim \varphi(q)$, and conversely.

Let $|E(f)| = n + 1$, $n \geqslant 3$. Then

$$\begin{aligned} f\, R_{n+1}\, g \quad &\text{iff} \quad f(0)\, R_n{}^*\, g(0) \\ f\, R_{n+1}^*\, g \quad &\text{iff} \quad f(1)\, R_n\, g(1) \end{aligned} \tag{2.20}$$

Proof. This is an immediate consequence of the definitions and (2.9).

III. The n-Skeleton, $n \geqslant 3$

An equation $(p = q)$ with $|E(p) \cup E(q)| = n$, $p \sim_n q$, $p \not\sim_{n+1} q$, will be called an equation in *n essential variables*. If $(p = q)$ is such an equation, there is an idempotent semigroup in which $(p = q)$ does not hold. However if φ is any primitive substitution in $(p = q)$ by less than n variables, then $(\varphi(p) = \varphi(q))$ holds in every idempotent semigroup.

Consider, for fixed n, the equational classes $[p = q]$, where $(p = q)$ is an equation in n essential variables. The subposet of the lattice of equational classes which consists of these elements will be called the *n-skeleton*. In this section the n-skeleton is completely described for $n \geqslant 3$.

The following notation is introduced (as were H^n, I^n, $\bar{I}^n$), to enable us to state and prove results for n even and n odd without separating the cases. If $f \in F(X)$, let

$$f^{*n} = \begin{cases} f & \text{if} \quad n \text{ is even} \\ f^* & \text{if} \quad n \text{ is odd.} \end{cases}$$

If $f_i \in F(X)$, $(i = 1, \ldots, r)$, let

$$\Big(\prod_{i=1}^{r} f_i\Big)_{*n} = \begin{cases} \prod_{i=1}^{r} f_i & \text{if} \quad n \text{ is even} \\ \Big(\prod_{i=1}^{r} f_i\Big)_* & \text{if} \quad n \text{ is odd.} \end{cases}$$

Lemma 3.1. *For $f, p \in F(X)$, assume $\hat{f}(\sigma_{n-2}) = (y_r y_{r-1} \cdots y_1)^{*n}$, $y_i \in X$, and let ψ be any substitution which satisfies the following properties*:

(i) $\psi(\bar{p}(\sigma_i)) = \begin{cases} \bar{f}(\sigma_i)\, \hat{f}(\sigma_{i+1}0) & \textit{if } i \textit{ is even} \\ \hat{f}(\sigma_{i+1}1)\, \bar{f}(\sigma_i) & \textit{if } i \textit{ is odd} \end{cases}$, *for all* $1 \leqslant i \leqslant n - 4$,

(ii) $\psi(\bar{p}(\sigma_{n-3})) = (y_u \cdots y_1 \bar{f}(\sigma_{n-3}))^{*n}$, *for some* $0 \leqslant u \leqslant r$,

(iii) $E(\psi(\hat{p}(\sigma_{n-2}))) = E(\hat{f}(\sigma_{n-2}))$.

Then the following is an expansion for $\bar{I}(f)$:

$$\hat{f}(00)\,\psi\left(\prod_{i=1}^{[1/2(n-2)]} p^{\#}(\sigma_{2i})\right)(\psi(\hat{p}(\sigma_{n-2}))\,\hat{f}(\sigma_{n-2}))_{*n}\,\psi\left(\left(\prod_{j=1}^{[1/2(n-3)]} p^{\#}(\sigma_{2j+1})\right)_{*}\bar{p}(0)\right).$$

Proof. It is an immediate consequence of Theorem 1.8 that if $Pf(\alpha)Q$ is an expansion for f, and if $E(h) \subseteq E(f(\alpha))$, then $P\hat{f}(\alpha 0)\,h\hat{f}(\alpha 1)Q$ is also an expansion for f. Using this result, and the successive expansions given in Lemma 2.12, we can prove Lemma 3.1 (in view of assumption (iii)), by establishing that $E(\psi(p(\sigma_i))) = E(f(\sigma_i))$, $1 \leqslant i \leqslant n-4$.

Using (1.15) and the definitions, it can be shown that, for $1 \leqslant i \leqslant n-4$,

$$\begin{aligned} E(\psi(p(\sigma_i))) &= \bigcup \{E(\psi(p(\sigma_j))) \mid j > i\} \\ &= \bigcup \{E(\bar{f}(\sigma_j)\,\hat{f}(\sigma_{j+1}0)) \mid i < j \leqslant n-4, j \text{ even}\} \\ &\quad \cup \bigcup \{E(\hat{f}(\sigma_{j+1}1)\,\bar{f}(\sigma_j)) \mid i < j \leqslant n-4, j \text{ odd}\} \\ &\quad \cup E(\bar{f}(\sigma_{n-3})) \cup E(\hat{f}(\sigma_{n-2})) \cup \{y_1, \ldots, y_u\} \\ &= \{\bar{f}(\sigma_j) \mid i < j\} = E(f(\sigma_i)). \end{aligned}$$

PROPOSITION 3.2. *Let* $|E(f)| = n \geqslant 3$, $f \sim_n g$, $p \sim_n q$, $p\not{R}_n q$. *Then* $[p = q] \subseteq [f = \bar{I}(g)\bar{F}(f)]$.

Proof. Since $p\not{R}_n q$, there exists a primitive substitution φ in $(p = q)$ by n variables such that $\varphi(p)\ \not{R}_n \varphi(q)$. It is therefore enough to prove the theorem if $|E(p)| = n$.

Since $p\not{R}_n q$ and $|E(p)| = n$, it follows that $I^{n+1}(\hat{p}(\sigma_{n-2})) \neq I^{n+1}(\hat{q}(\sigma_{n-2}))$. Moreover, Proposition 2.13 (iii) states that $H^{n+1}(\hat{p}(\sigma_{n-2})) = H^{n+1}(\hat{q}(\sigma_{n-2}))$. Therefore, we may assume that $\hat{p}(\sigma_{n-2}) = (p_1bc)_{*n}$ and $\hat{q}(\sigma_{n-2}) = (q_1ac)_{*n}$ for some a, b, $c \in X$ and $p_1, q_1 \in F(X)$ with $E(p_1) \cup E(q_1) \subseteq \{a, b, c\}$. Let $\hat{f}(\sigma_{n-2}) = (y_r \cdots y_1)^{*n}$ and $\hat{g}(\sigma_{n-2}) = (z_s \cdots z_1)^{*n}$, where $y_i, z_i \in X$. From Proposition 2.13 (iii) it follows that $y_1 = z_1$. It follows (since $\hat{f}(\sigma_{n-2}) = \bar{I}^{n+1}(\hat{f}(\sigma_{n-2}))$), that if $\hat{f}(\sigma_{n-2}) \neq \hat{g}(\sigma_{n-2})$ then there exists a t such that $y_j = z_j$, $1 \leqslant j < t$ and $y_t \neq z_t$.

Let ψ be the substitution referred to in Lemma 3.1, satisfying the following additional conditions:

$$\begin{aligned} \psi(c) &= (y_{t-1} \cdots y_1)^{*n} = (z_{t-1} \cdots z_1)^{*n}, \\ \psi(b) &= (y_r \cdots y_t)^{*n}, \\ \psi(a) &= z_t. \end{aligned}$$

A direct calculation from these definitions shows that

$$(\psi(\hat{p}(\sigma_{n-2}))\,\hat{f}(\sigma_{n-2}))_{*n} \sim \psi(\hat{p}(\sigma_{n-2})).$$

Combining this with Lemma 3.1, we have shown that $f \sim \hat{f}(00)\psi(A_n(p))\bar{F}(f)$. It follows therefore that $[p = q] \subseteq [\psi(p) = \psi(q)] \subseteq [f = \hat{f}(00)\,\psi(A_n(q))\bar{F}(f)]$.

Define f_1 by replacing $\hat{f}(\sigma_{n-2})$ with $\psi(\hat{q}(\sigma_{n-2}))$ in $A_n(f)$. Then it can be shown that $f_1(\alpha) = f(\alpha)$ and $\bar{f}_1(\alpha) = \bar{f}(\alpha)$ for all $f(\alpha)$, $\bar{f}(\alpha)$ occurring in $A_n(f)$ with $\alpha \neq \sigma_{n-2}$, and that $\hat{f}_1(\sigma_{n-2}) = \bar{I}^{n+1}(\psi(\hat{q}(\sigma_{n-2})))$. A direct calculation then shows that $\hat{f}_1(\sigma_{n-2}) = \bar{I}^{n+1}((\psi(q_1)\, z_t \cdots z_1)_{*n})$. Also, by Lemma 3.1, $f_1 \sim \hat{f}(00)\, \psi(A_n(q))\, \bar{F}(f)$. Therefore, we have in fact shown that $[p = q] \subseteq [f = f_1]$.

The above considerations can be summarized by saying that we have found from f an f_1 such that (with the definitions of $\hat{f}$ (σ_{n-2}) and $\hat{g}(\sigma_{n-2})$ given above), if $\hat{f}_1(\sigma_{n-2}) = (y_{r'} \cdots y_1')^{*n}$, then $y_j' = z_j$ for $0 \leqslant j \leqslant t$. By induction (on $s - t$) it follows therefore that $[p = q] \subseteq [f_1 = \bar{I}(g)\bar{F}(f)]$, and hence that $[p = q] \subseteq [f = \bar{I}(g)\bar{F}(f)]$.

The dual of Proposition 3.2 is the statement with p, q, f, g replaced everywhere with p^*, q^*, f^*, g^*, respectively. From the various definitions involved it is routine to check that this is equivalent to

PROPOSITION 3.2*. *Let* $|E(f)| = n \geqslant 3$, $f \sim_n g$, $p \sim_n q$, pR_n^*q. *Then* $[p = q] \subseteq [f = \bar{I}(f)\bar{F}(g)]$.

PROPOSITION 3.3. *Let* $|E(f)| = n \geqslant 3$, $f \sim_n g$, $fR_n g$, $p \sim_n q$, $p\mathcal{S}_n q$. *Then* $[p = q] \subseteq [f = \bar{I}(g)\bar{F}(f)]$.

Proof. By an argument similar to that used in the proof of Proposition 3.2, it is enough to prove Proposition 3.3 in case $|E(p)| = n$. Moreover, by Proposition 3.2 we need only consider the case $pR_n q$. It follows that $I^{n+1}(\hat{p}(\sigma_{n-2})) = I^{n+1}(\hat{q}(\sigma_{n-2}))$ and $\hat{p}(\sigma_{n-2}) \not\sim \hat{q}(\sigma_{n-2})$. Therefore, we may assume $\hat{p}(\sigma_{n-2}) = (acbc)^{*n}$ and $\hat{q}(\sigma_{n-2}) = (abc)^{*n}$ for some a, b, $c \in X$.

Let $\hat{f}(\sigma_{n-2}) = (y_r \cdots y_1)^{*n}$, and assume that $y_t = y_j$ for some t, j with $j < t - 1$. Let ψ be the substitution referred to in Lemma 3.1, satisfying the additional conditions $u = j - 1$, $\psi(c) = y_j = y_t$, $\psi(b) = (y_{t-1} \cdots y_{j+1})^{*n}$, and $\psi(a) = (y_r \cdots y_{t+1})^{*n}$. It is a routine calculation to check that

$$\psi(p^{\#}(\sigma_{2[1/2(n-2)]}))(\psi(p(\sigma_{n-2}))\, \hat{f}(\sigma_{n-2}))_{*n}\, \psi(p^{\#}(\sigma_{2[1/2(n-3)]+1}))$$
$$\sim \psi(p^{\#}(\sigma_{2[1/2(n-2)]}))\, \psi(p(\sigma_{n-2}))\, \psi(p^{\#}(\sigma_{2[1/2(n-3)]+1})).$$

Therefore Lemma 3.1 implies that $f \sim \hat{f}(00)\, \psi(A_n(p))\, \bar{F}(f)$ and hence $[p = q] \subseteq [f = \hat{f}(00)\, \psi(A_n(q))\, \bar{F}(f)]$.

Define f_1 by replacing $\hat{f}(\sigma_{n-2})$ with $(\psi(\hat{q}(\sigma_{n-2}))\, y_{j-1} \cdots y_1)_{*n}$ in $A_n(f)$. It can be shown that $f_1(\alpha) = f(\alpha)$ and $\bar{f}_1(\alpha) = \bar{f}(\alpha)$ for $f(\alpha)$, $\bar{f}(\alpha)$ occurring in $A_n(f)$, $\alpha \neq \sigma_{n-2}$. Lemma 3.1 immediately gives that $f_1 \sim \hat{f}(00)\psi(A_n(q))\bar{F}(f)$, and it follows that $[p = q] \subseteq [f = f_1]$. A direct calculation shows that $f_1(\sigma_{n-2}) \sim (y_r \cdots y_{t+1} y_{t-1} \cdots y_1)^{*n}$.

The above results can be summarized by saying that from f we may find an f_1 such that $[p = q] \subseteq [f = f_1]$, $f_1(\alpha) = f(\alpha)$, $\bar{f}_1(\alpha) = \bar{f}(\alpha)$ for all $f(\alpha)$ and

$f(\alpha)$ occurring in $A_n(f)$ with $\alpha \neq \sigma_{n-2}$, $I^{n+1}(\hat{f}(\sigma_{n-2})) = I^{n+1}(\hat{f}_1(\sigma_{n-2}))$ and $L(\hat{f}_1(\sigma_{n-2})) < L(\hat{f}(\sigma_{n-2}))$. By induction on $L(f(\sigma_{n-2}))$, we can therefore conclude that there is an h with the same properties as f_1, and in addition such that $\hat{h}(\sigma_{n-2}) = I^{n+1}(\hat{f}(\sigma_{n-2}))$. Similarly, there is an h_1 such that $[p = q] \subseteq [g = h_1]$ and $g(\alpha) = h_1(\alpha)$, $\bar{g}(\alpha) = \bar{h}_1(\alpha)$ for all $g(\alpha)$, $\bar{g}(\alpha)$ occurring in $A_n(g)$, $\alpha \neq \sigma_{n-2}$, and such that $\hat{h}_1(\sigma_{n-2}) = I^{n+1}(\hat{g}(\sigma_{n-2})) = I^{n+1}(\hat{f}(\sigma_{n-2}))$. From Corollary 2.14 (iii), it follows that $h \sim \bar{I}(h_1)\bar{F}(f)$ and therefore $[p = q] \subseteq [f = \bar{I}(g)\bar{F}(f)]$.

The dual of Proposition 3.3 can be shown to be equivalent to

PROPOSITION 3.3*. *Let* $|E(f)| = n \geq 3$, $f \sim_n g$, $f\, R_n^*\, g$, $p \sim_n q$, $p \not{S}_n^*\, q$. *Then* $[p = q] \subseteq [f = \bar{I}(f)\bar{F}(g)]$.

THEOREM 3.4. *Let* $|E(f)| = n \geq 3$, $p \sim_n q$, $f \sim_n g$. *Then any of the following eight conditions is sufficient for* $[p = q] \subseteq [f = g]$.

(1) $p \not{R}_n q$, $\quad p \not{R}_n^* q$

(2) $p \not{S}_n q$, $\quad p \not{R}_n^* q$, $\quad f R_n g$

(3) $p \not{S}_n^* q$, $\quad p \not{R}_n q$, $\quad f R_n^* g$

(4) $p \not{R}_n^* q$, $\quad f S_n g$

(5) $p \not{R}_n q$, $\quad f S_n^* g$

(6) $p \not{S}_n q$, $\quad p \not{S}_n^* q$, $\quad f R_n g$, $\quad f R_n^* g$

(7) $p \not{S}_n^* q$, $\quad f S_n g$, $\quad f R_n^* g$

(8) $p \not{S}_n q$, $\quad f S_n^* g$, $\quad f R_n g$

Proof. By Lemma 1.23, it is sufficient to show in each case that the given conditions imply that $(f = \bar{I}(f)\bar{F}(g))$ and $(g = \bar{I}(f)\bar{F}(g))$ hold if $(p = q)$ holds. This can be done by application of Corollary 2.14 and Propositions 3.2, 3.2*, 3.3, and 3.3*. By way of illustration we will prove (1) and (4), and leave the others to the reader.

If $p \not{R}_n q$, then by 3.2, $[p = q] \subseteq [g = \bar{I}(f)\bar{F}(g)]$. If $p \not{R}_n^* q$, then by 3.2* $[p = q] \subseteq [f = \bar{I}(f)\bar{F}(g)]$. Therefore (1) is proved.

Assuming again that $p \not{R}_n^* q$, we have $[p = q] \subseteq [f = \bar{I}(f)\bar{F}(g)]$. Moreover if $f S_n g$, it follows by Corollary 2.14 that $\bar{I}(f) \sim \bar{I}(g)$ and therefore that $(g = \bar{I}(f)\bar{F}(g))$ holds. Thus (4) is proved.

COROLLARY 3.5. *For any* $n \geq 3$, *the n-skeleton consists of at most eight elements.*

Proof. This follows from Theorem 3.4 and statements (2.17) and (2.18). In fact the set of equations in n essential variables ($n \geq 3$) is partitioned by R_n, R_n^*, S_n and S_n^* into eight sets in such a way that if two equations belong to the same set, they determine the same equational class.

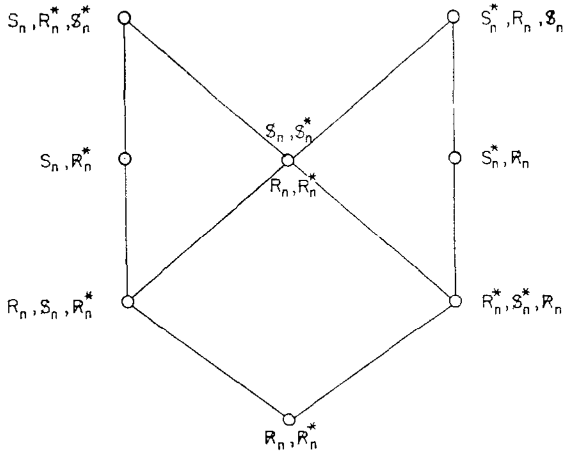

FIG. 1. The n-skeleton, $n \geqslant 3$.

Theorem 3.4 is summarized in Fig. 1. We next prove a series of lemmas which will show that all nontrivial inclusions among elements of the n-skeleton (for fixed n) have been given in Fig. 1. In particular, we will have shown that the eight classes are distinct elements of the lattice of equational classes of idempotent semigroups.

We first show that $\sim_n$, which by definition is preserved by primitive substitution, is preserved even by arbitrary substitution.

LEMMA 3.6. *For any substitution* ψ, *if* $p \sim_n q$ *then* $\psi(p) \sim_n \psi(q)$. *In particular if* $|E(\psi(p)) \cup E(\psi(q))| < n$, *then* $\psi(p) \sim \psi(q)$.

Proof. We have to show that if φ is a primitive substitution in $(\psi(p) = \psi(q))$ by less than n variables, then $\varphi(\psi(p)) \sim \varphi(\psi(q))$. But $\varphi \circ \psi$ is a substitution and therefore it is enough to show that if

$$|E(\psi(p)) \cup E(\psi(q))| < n \qquad \text{then} \quad \psi(p) \sim \psi(q).$$

We will show that $\bar{I}(\psi(p)) \sim \bar{I}(\psi(q))$. The remainder of the proof is dual.

For $n = 2$, the result is trivial. If $n = 3$, then since $p \sim_3 q$, it follows that $H(p) = H(q)$, and $E(p) = E(q)$. Therefore by (1.17) and (1.6), $H(\psi(p)) = H(\psi(q))$ and $E(\psi(p)) = E(\psi(q))$. Since $|E(\psi(p))| < 3$, it follows that $\bar{I}(\psi(p)) \sim \bar{I}(\psi(q))$.

We continue by induction on $n \geqslant 4$. There exists an $r \geqslant 1$ with $E(\psi(\hat{p}(0^r))) = E(\psi(p)) \neq E(\psi(p(0^r)))$. By Proposition 2.5, $p(0^r) \sim_{n-1} q(0^r)$, and since $|E(\psi(p(0^r)))| < n - 2$, it follows by induction that $\psi(p(0^r)) \sim \psi(q(0^r))$. Moreover, $\bar{p}(0^r) = \bar{q}(0^r)$, and therefore $\psi(\hat{p}(0^r)) \sim \psi(\hat{q}(0^r))$. By definition of r, there exist $w, w' \in F'(X)$ such that $\bar{I}(\psi(p))w = \psi(\hat{p}(0^r)) \sim \psi(\hat{q}(0^r)) = \bar{I}(\psi(q))w'$, and hence $\bar{I}(\psi(p)) \sim \bar{I}(\psi(q))$.

(3.7) $[p = q] \subseteq [f = g]$ if, and only if, there exist $h_0, h_1, \ldots, h_n \in F(X)$ such that $h_0 = f$, $h_n = g$ and such that for each $0 \leqslant i < n$ there exist $u_i, v_i \in F'(X)$, and substitutions ψ_i, with $h_i = u_i\psi_i(p)v_i$ and $h_{i+1} = u_i\psi_i(q)v_i$, or $h_i = u_i\psi_i(q)v_i$ and $h_{i+1} = u_i\psi_i(p)v_i$.

Statements (3.7) and (1.1) are both special cases of the same lemma of Universal Algebra.

Proposition 3.8. *Assume $f \sim_n g$, $p \sim_n q$, $| E(p)| = | E(f)| = n \geqslant 3$. Then either of the following implies that $[p = q] \nsubseteq [f = g]$:*

(i) $p \, R_n \, q, \qquad f \, \not{R}_n \, g$

(ii) $p \, S_n \, q, \qquad f \, \not{S}_n \, g$

Proof. Let $f \, \not{R}_n \, g$. Then from the definition of R_n, and Proposition 2.13, it follows that $\hat{f}(\sigma_{n-2})$ and $\hat{g}(\sigma_{n-2})$ satisfy the following properties:

$$I^{n+1}(\hat{f}(\sigma_{n-2})) \neq I^{n+1}(\hat{g}(\sigma_{n-2})), \quad E(\hat{f}(\sigma_{n-2})) = E(\hat{g}(\sigma_{n-2}))$$

and

$$H^{n+1}(\hat{f}(\sigma_{n-2})) = H^{n+1}(\hat{g}(\sigma_{n-2})).$$

Also, by (2.8), $| E(\hat{f}(\sigma_{n-2}))| = 3$. It follows that $H^n(\hat{f}(\sigma_{n-2})) \neq H^n(\hat{g}(\sigma_{n-2}))$. Therefore by (3.7) it is enough to show that if $f = u\psi(p)v$ and $g = u\psi(q)v$ for some $u, v \in F'(X)$ and substitution ψ, then $H^n(\hat{f}(\sigma_{n-2})) = H^n(\hat{g}(\sigma_{n-2}))$.

By Proposition 2.13 we can assume without loss of generality that $p(\alpha) = q(\alpha)$ and $\bar{p}(\alpha) = \bar{q}(\alpha)$ for all $p(\alpha)$, $\bar{p}(\alpha)$ occurring in $A_n(p)$, $\alpha \neq \sigma_{n-2}$, $\alpha \neq \sigma'_{n-2}$. Moreover since $p \, R_n \, q$ it follows that $I^{n+1}(\hat{p}(\sigma_{n-2})) = I^{n+1}(\hat{q}(\sigma_{n-2}))$ and therefore in particular that $H^n(\hat{p}(\sigma_{n-2})) = H^n(\hat{q}(\sigma_{n-2}))$. Let $u\psi(p) = (x_1x_2 \cdots x_r)^{*n}$ and $u\psi(q) = (y_1y_2 \cdots y_s)^{*n}$. It follows that if $H^n((\psi(p))(\sigma_{n-2})) = x_j$ then $x_k = y_k$ for all $k \leqslant j$, and that $H^n((\psi(q))(\sigma_{n-2})) = y_j = x_j$. Now let $H^n((u\psi(p))(\sigma_{n-2})) = x_i$. Then $i \leqslant j$, and therefore $H^n(\hat{f}(\sigma_{n-2})) = H^n(\hat{g}(\sigma_{n-2}))$.

To show part (ii) we proceed in a similar way. Since $f \, \not{S}_n \, g$, it follows that $\bar{I}(f) \nsim \bar{I}(g)$ and therefore by (3.7) it is enough to show that if $f = u\psi(p)v$ and $g = u\psi(q)v$, for some $u, v \in F'(X)$ and some substitution ψ, then $\bar{I}(f) \sim \bar{I}(g)$.

Since $| E(f)| = n$, $| E(\psi(p))| \leqslant n$. If $| E(\psi(p))| < n$ then by Lemma 3.6 $\psi(p) \sim \psi(q)$. If $| E(\psi(p))| = n$ then since $p \, S_n \, q$ it follows that $\bar{I}(\psi(p)) \sim \bar{I}(\psi(q))$. In either case, we can conclude that $\bar{I}(f) \sim \bar{I}(g)$.

Theorem 3.9. *Figure 1 represents the n-skeleton of the lattice of equational classes of idempotent semigroups, for all $n \geqslant 3$.*

Proof. Proposition 3.8 and the dual statements show that all nontrivial inclusions among the elements in Figure 1 have been given in Figure 1, that is by Theorem 3.4.

IV. The Lattice

In this last section we describe all the inclusions which exist among the equational classes determined by one equation, and show that every equational class of idempotent semigroups is determined by a single equation. The first step is to generalize Propositions 3.2 and 3.3 by removing the cardinality restriction on $E(f)$. This is done in part 1.

The union of the n-skeletons for all $n \geqslant 2$, together with the restriction of the order in the lattice of equational classes, will be called the *skeleton* of the lattice. We have previously described the n-skeleton for $n \geqslant 3$. In part 2, we describe the inclusions which exist among the elements of different n-skeletons ($n \geqslant 3$). A description of the 2-skeleton is given as part of Propositions 4.30, and the inclusions between the elements of the 2-skeleton and the rest of the skeleton are given in Proposition 4.31. Thus the skeleton of the lattice is completely described.

It remains to relate the equational classes determined by a single non-skeletal equation to the skeleton. Propositions 4.1 and 4.20 give some results of this kind. In part 3 we prove some further results, and in particular introduce the relations T_n and $T_n{}^*$ which together with the relations previously introduced completely characterize the conditions under which two equations determine the same equational class. These results are summarized in Propositions 4.29 and 4.30.

In part 4 the description of the lattice is completed by showing that every equational class of idempotent semigroups is determined by a single equation. The complete lattice is depicted in Fig. 2.

1. *Generalization of Propositions* 3.2 *and* 3.3

Proposition 4.1 (Generalization of Proposition 3.2). *Let* $n \geqslant 3$, $p \sim_n q$, $p \, R_n \, q$, $f \sim_n q$. *Then* $[p = q] \subseteq [f = \bar{I}(g)\bar{F}(f)]$.

Proof. In the proof of Proposition 3.2, the cardinality restriction on f was used only at the last step. In fact if g' is defined by replacing $\hat{g}(\sigma_{n-2})$ with $\hat{f}(\sigma_{n-2})$ in $A_n(g)$, the cardinality restriction was used essentially to show that $f \sim \bar{I}(g')\bar{F}(f)$, via Corollary 2.14. In order to prove Proposition 4.1, it is therefore enough to prove

$$[p = q] \subseteq [f = \bar{I}(g')\bar{F}(f)]. \tag{4.2}$$

Assume $|E(f)| = n + k$ where without loss of generality $k \geqslant 0$. The proof is by induction on k. If $k = 0$, (4.2) is trivial, for in this case $f \sim \bar{I}(g')\bar{F}(f)$, as was noted above. Assume that (4.2) holds for all f with $|E(f)| = n + k - 1$, some $k \geqslant 1$.

Let r be an arbitrary but fixed integer with $0 \leqslant r \leqslant [\frac{1}{2}(n-4)]$ and consider the primitive substitution φ which satisfies the following conditions:

$$\varphi(\bar{f}(\sigma_{2r+1})) = \bar{f}(\sigma_{2r+1}0)$$
$$\varphi(x) = x \qquad \text{for all} \quad x \neq \bar{f}(\sigma_{2r+1}).$$

Then φ is a primitive substitution in $(f = g)$ by $(n + k - 1)$ variables.

From the definition of φ it follows that $\varphi(\hat{f}(\sigma_{2r+1}0)) = \hat{f}(\sigma_{2r+1}0)$, and $E(\varphi(\hat{f}(\sigma_{2r+1}0))) = E(\varphi(\hat{f}(\sigma_{2r}1))) = E(\hat{f}(\sigma_{2r+2})\bar{f}(\sigma_{2r+1}))$. Therefore (by Theorem 1.8) it follows that

$$\begin{aligned}\varphi(f^{\#}(\sigma_{2r+2})f(\sigma_{2r+2})f^{\#}(\sigma_{2r+1})) &\sim \hat{f}(\sigma_{2r+1}0)\,\varphi(\hat{f}(\sigma_{2r}1)) \\ &= f(\sigma_{2r+1}0)\,\varphi(f^{\#}(\sigma_{2r+1})).\end{aligned} \tag{4.3}$$

Consider the following two expressions: Lemma 2.12 (i) with f replaced by $\varphi(f)$; and φ applied to the expansion for f given by Lemma 2.12 (ii), for the fixed r given above. By comparing these two expansions (which are both in the relation $\sim$ to $\varphi(f)$, and are therefore in the relation $\sim$ to each other), and using (4.3), it can be shown that

$$\varphi(f)(\sigma_{2r+1}) \sim f(\sigma_{2r+1}0). \tag{4.4}$$

Consider a substitution ψ with the following properties:

$$\begin{aligned}\psi(x) &= x, &&\text{for all} \quad x \in E(\varphi(f)(\sigma_{2r+1})) \\ \psi(\overline{\varphi(f)}(\sigma_{2i})) &= \bar{f}(\sigma_{2i})\,\hat{f}(\sigma_{2i+1}0), && 1 \leqslant i \leqslant r-1 \\ \psi(\overline{\varphi(f)}(\sigma_{2j+1})) &= \hat{f}(\sigma_{2j+2}1)\,\bar{f}(\sigma_{2j+1}), && 0 \leqslant j \leqslant r-1 \\ \psi(\overline{\varphi(f)}(\sigma_{2r})) &= \bar{f}(\sigma_{2r}) \\ \psi(\overline{\varphi(f)}(\sigma_{2r+1})) &= f^{\#}(\sigma_{2r+1}0)\,\bar{f}(\sigma_{2r+1}).\end{aligned} \tag{4.5}$$

From these definitions and (4.4) it follows that

$$\psi(\varphi(f)(\sigma_{2r+1})) \sim f(\sigma_{2r+1}0). \tag{4.6}$$

Moreover, by a method similar to that used in the proof of Lemma 3.1, we can show that

$$E(\psi((\varphi(f))^{\wedge}(\sigma_{2i+1}0))) = E(\hat{f}(\sigma_{2i+1}0)), \qquad 0 \leqslant i \leqslant r-1,$$

and

$$E(\psi((\varphi(f))^{\wedge}(\sigma_{2j}1))) = E(\hat{f}(\sigma_{2j}1)), \qquad 1 \leqslant j \leqslant r. \tag{4.7}$$

From (4.6) and (4.7) it now follows, by the same method of proof as in Lemma 3.1, that

$$f \sim \hat{f}(00)\,\psi(\varphi(f))\,\bar{F}(f). \tag{4.8}$$

Now $|E(\psi(\varphi(f)))| < n + k$, and therefore by inductive hypothesis it follows that $[p = q] \subseteq [\psi(\varphi(f)) = \bar{I}(\psi(\varphi(g)))\,\bar{F}(\psi(\varphi(f)))]$. Therefore

$$[p = q] \subseteq [f = \hat{f}(00)\,\psi(\varphi(g))\,\bar{F}(f)], \tag{4.9}$$

since $\hat{f}(00)\,\psi(\varphi(g))\,\bar{F}(f) \sim \hat{f}(00)\,\bar{I}(\psi(\varphi(g)))\,\bar{F}(\psi(\varphi(f)))\,\bar{F}(f)$.

Since $\varphi(f) \sim_n \varphi(g)$, it follows from Corollary 2.6 and Proposition 2.5 that

$$\overline{\varphi(f)}(\sigma_i) = \overline{\varphi(g)}(\sigma_i), \qquad 1 \leqslant i \leqslant 2r + 1 \tag{4.10}$$

(since $r \leqslant [\frac{1}{2}(n - 4)]$). Moreover by Proposition 2.4, $E(\varphi(f)(\sigma_{2r+1})) = E(\varphi(g)(\sigma_{2r+1}))$. It follows therefore that (4.5) holds with f replaced by g.

Consider g_r, defined by replacing $f(\sigma_{2r+1}0)$ with $g(\sigma_{2r+1}0)$ in the expansion for f given by Lemma 2.12 (i) (with $f(\sigma_{2r+1})$ replaced by $\hat{f}(\sigma_{2r+1}0)\,\underline{f}(\sigma_{2r+1}1)$). Since we can conclude from (4.10) that $\psi(\overline{\varphi(g)}(\sigma_i)) = \psi(\overline{\varphi(f)}(\sigma_i))$, $1 \leqslant i \leqslant 2r + 1$, and since we can also establish that (4.6) and (4.7) hold with f replaced by g, it can be shown by a method similar to the one used to establish (4.8) that

$$g_r \sim \hat{f}(00)\,\psi(\varphi(g))\,\bar{F}(f). \tag{4.11}$$

Combining this with (4.9), we can finally conclude that, for any $0 \leqslant r \leqslant [\frac{1}{2}(n - 4)]$,

$$[p = q] \subseteq \left[f = \left(\prod_{i=1}^{r} f^{\neq}(\sigma_{2i})\right) g(\sigma_{2r+1}0)\,f^{\neq}(\sigma_{2r+1}0)\left(\prod_{j=0}^{r} f^{\neq}(\sigma_{2j+1})\right)_{*}\right]. \tag{4.12}$$

Following the pattern of proof leading to statement (4.12), we now let φ be a substitution with the properties

$$\varphi(\bar{f}(\sigma_{2s})) = \bar{f}(\sigma_{2s}1)$$
$$\varphi(x) = x \qquad \text{for all} \quad x \neq \bar{f}(\sigma_{2s}),$$

where s is an arbitrary but fixed integer with $1 \leqslant s \leqslant [\frac{1}{2}(n - 3)]$. In the same way as we proved (4.4), we can now establish

$$\varphi(f)(\sigma_{2s}) \sim f(\sigma_{2s}1). \tag{4.13}$$

Consider a substitution ψ with the following properties:

$$
\begin{aligned}
\psi(x) &= x \qquad \text{for all} \quad x \in E(\varphi(f)(\sigma_{2s})) \\
\psi(\overline{\varphi(f)}(\sigma_{2i})) &= \bar{f}(\sigma_{2i})\, \dot{f}(\sigma_{2i-1}0), \qquad 1 \leqslant i \leqslant s-1 \\
\psi(\overline{\varphi(f)}(\sigma_{2j+1})) &= \hat{f}(\sigma_{2j+2}1)\, \bar{f}(\sigma_{2j+1}), \qquad 0 \leqslant j \leqslant s-2 \\
\psi(\overline{\varphi(f)}(\sigma_{2s-1})) &= \bar{f}(\sigma_{2s-1}) \\
\psi(\overline{\varphi(f)}(\sigma_{2s})) &= \bar{f}(\sigma_{2s})\, f^{\#}(\sigma_{2s}1).
\end{aligned}
\tag{4.14}
$$

Using (4.13) and (4.14), and the pattern of proof of (4.12), we can finally show that, for any $1 \leqslant s \leqslant [\frac{1}{2}(n-3)]$,

$$[p = q] \subseteq \left[f = \left(\prod_{i=1}^{s} f^{\#}(\sigma_{2i})\right) f^{\#}(\sigma_{2s}1)\, g(\sigma_{2s}1) \left(\prod_{j=0}^{s-1} f^{\#}(\sigma_{2j+1})\right)_{*}\right]. \tag{4.15}$$

Since $f \sim_n g$ it follows from Corollary 2.6 that $f(\sigma_t) \sim_4 g(\sigma_t)$, $1 \leqslant t \leqslant n-4$, and therefore by Proposition 2.5 that

$$\bar{f}(\sigma_t) = \bar{g}(\sigma_t), \qquad 1 \leqslant t \leqslant n-3 \tag{4.16}$$

$$\bar{f}(\sigma_t 0) = \bar{g}(\sigma_t 0), \quad 1 \leqslant t \leqslant n-3, \ t \text{ odd} \tag{4.17}$$

$$\bar{f}(\sigma_t 1) = \bar{g}(\sigma_t 1), \quad 1 \leqslant t \leqslant n-3, \ t \text{ even}. \tag{4.18}$$

Now (4.2) can be derived from (4.12), (4.15), (4.16), (4.17), and (4.18) as follows.

In $A_n(f)$ we can replace $\bar{f}(\alpha)$ by $\bar{g}(\alpha)$ for all the α for which we have proved that $\bar{f}(\alpha) = \bar{g}(\alpha)$ (see 4.16, 4.17, 4.18). In (4.12) and (4.15) we have proved that if $(p = q)$ holds, we can replace $f(\alpha)$ by $g(\alpha)$ in $A_n(f)$ for any one of the sequences $\alpha = \sigma_{2r+1}0$ (for $0 \leqslant r \leqslant [\frac{1}{2}(n-4)]$) or $\alpha = \sigma_{2s}1$ (for $1 \leqslant s \leqslant [\frac{1}{2}(n-3)]$). To establish (4.2) we must show that in fact we can replace $f(\alpha)$ by $g(\alpha)$ for all these α at once. Now the result of making one such replacement is a term f^1, say, which can be shown to satisfy the properties of f which were used to establish (4.12) and (4.15). We can therefore make a second (different) replacement in f^1, to obtain f^2, and so on until we finally arrive at a term which is in the relation $\sim$ to $\bar{I}(g')\bar{F}(f)$. This establishes (4.2) and hence the proposition.

LEMMA 4.19. *Let* $n \geqslant 3$, $f \sim_n g$ *and* $f R_n g$. *Then* $I^{n+1}(\hat{f}(\sigma_{n-2})) = I^{n+1}(\hat{g}(\sigma_{n-2}))$.

Proof. Assume $I^{n+1}(\hat{f}(\sigma_{n-2})) \neq I^{n+1}(\hat{g}(\sigma_{n-2}))$ for some f, g with $f \sim_n g$ and $f R_n g$. By (1.22), there exists a primitive substitution φ in $(\hat{f}(\sigma_{n-2}) = \hat{g}(\sigma_{n-2}))$ by 3 variables such that

$$I^{n+1}(\varphi(\hat{f}(\sigma_{n-2}))) \neq I^{n+1}(\varphi(\hat{g}(\sigma_{n-2}))).$$

We can, of course, assume that $\varphi(x) = x$ for all $x \notin E(\hat{f}(\sigma_{n-2}))$. Then φ is a primitive substitution in $(f = g)$ by n variables.

Now $\varphi(f) \sim_n \varphi(g)$, $\varphi(f)\, R_n\, \varphi(g)$, and $|E(\varphi(f))| = n$, and therefore $I^{n+1}((\varphi(f))^\frown(\sigma_{n-2})) = I^{n+1}((\varphi(g))^\frown(\sigma_{n-2}))$. If we can show that

$$(\varphi(f))^\frown(\sigma_{n-2}) \sim \varphi(\hat{f}(\sigma_{n-2})),$$

we will have arrived at the desired contradiction.

If $\bar{f}(0) \notin E(\hat{f}(\sigma_{n-2}))$, (i.e., if $n \geqslant 3$) it follows that $\varphi(\bar{f}(0)) = f(0)$, and $\varphi(x) \neq \bar{f}(0)$ if $x \neq \bar{f}(0)$, and therefore $\varphi(f(0)) = \varphi(f))(0)$, (by (1.19)). Assume $\varphi(f(\sigma_i)) = (\varphi(f))(\sigma_i)$ for some i. Then if $\bar{f}(\sigma_{i+1}) \notin E(\hat{f}(\sigma_{n-2}))$ it follows that $\varphi(f(\sigma_{i+1})) = (\varphi(f))(\sigma_{i+1})$. Therefore we can conclude by induction that $\varphi(f(\sigma_{n-3})) = (\varphi(f))(\sigma_{n-3})$. But then (for any $n \geqslant 3$),

$$\begin{aligned}(\varphi(f))^\frown(\sigma_{n-2}) &= \bar{I}^{n+1}(\varphi(f)(\sigma_{n-3})) \sim \bar{I}^{n+1}(\varphi(f(\sigma_{n-3}))) \sim \bar{I}^{n+1}(\varphi(\bar{I}^{n+1}(f(\sigma_{n-3}))))\\ &= \bar{I}^{n+1}(\varphi(\hat{f}(\sigma_{n-2}))) = \varphi(\hat{f}(\sigma_{n-2})).\end{aligned}$$

Proposition 4.20 (Generalization of Proposition 3.3). *Let $n \geqslant 3$, $p \sim_n q$, $p\, \mathcal{S}_n\, q$, $f \sim_n g$, $f\, R_n\, g$. Then $[p = q] \subseteq [f = \bar{I}(g)\bar{F}(f)]$.*

Proof. (Compare the proof of Proposition 4.1). In the proof of Proposition 3.3, the cardinality restriction $|E(f)| = n$ made Lemma 4.19 and statement (4.2) trivial, and moreover this was the only use made of this restriction. It is therefore (as in the proof of Proposition 4.1), enough to prove (4.2) under the assumptions of Proposition 4.20. This proof is essentially the same as the proof of (4.2) in Proposition 4.1. In fact, if $f\, R_n\, g$, then $\varphi(f)\, R_n\, \varphi(g)$ for any primitive substitution by $|E(f)| - 1$ variables, giving the inductive step.

2. *The Skeleton of the Lattice*

We are now in a position to describe the inclusions which exist among the elements of different n-skeletons for $n \geqslant 3$.

Proposition 4.21. *Let* $|E(p)| = n \geqslant 3$, $p \sim_n q$, $|E(f)| = n + 1$, $f \sim_{n+1} g$. *Then each of the following conditions is sufficient for* $[p = q] \subseteq [f = g]$.

(i) $p\, \mathcal{S}_n\, q, \quad f\, R^*_{n+1}\, g$

(ii) $p\, \mathcal{R}_n\, g$

(iii) $p\, \mathcal{S}_n\, q, \quad p\, \mathcal{S}_n{}^*\, q$

Proof. (i) Since $f\, R^*_{n+1}\, g$, it follows from (2.20) that $f(1)\, R_n\, g(1)$ and therefore by Proposition 3.3 that $[p = q] \subseteq [f(1) = \bar{I}(g(1))\bar{F}(f(1))]$. But $f \sim_{n+1} g$ and $|E(f)| = n + 1 \geqslant 4$, and therefore by Proposition 2.13,

$f(11) \sim g(11)$, $\bar{f}(11) = \bar{g}(11)$, and $f(1) = \bar{g}(1)$. It follows that $[p = q] \subsetneq [f = \bar{I}(f)\bar{F}(g)]$.

Since $f \sim_{n+1} g$, it follows by (2.19) and (2.18) that $f\ R_n\ g$. Therefore by Proposition 4.20, $[p = q] \subsetneq [f = \bar{I}(g)\bar{F}(f)]$.

(ii) From Proposition 4.1, it follows immediately that $[p = q] \subsetneq [f = \bar{I}(g)\bar{F}(f)]$, and since $f(1) \sim_n g(1)$, that

$$[p = q] \subseteq [f(1) = \bar{I}(g(1))\bar{F}(f(1))].$$

As in the proof of (i), $\bar{F}(f(1)) \sim \bar{F}(g(1))$ and $\bar{f}(1) = \bar{g}(1)$, and therefore it follows that $[p = q] \subseteq [f = g]$.

(iii) Let f_i, g_i, $(i = 1, 2, 3)$, satisfy $f_i \sim_{n+1} g_i$, $|E(f_i)| = n + 1$. Let $f_1\, R^*_{n+1}\, g_1$, $f_1\, \cancel{R}_{n+1}\, g_1$; let $f_2\, \cancel{R}^*_{n+1}\, g_2$, $f_2\, R_{n+1}\, g_2$; and let $f_3\, \cancel{R}^*_{n+1}\, g_3$, $f_3\, \cancel{R}^*_{n+1}\, g_3$. We then have by (i) and its dual that $[p = q] \subseteq [f_1 = g_1] \wedge [f_2 = g_2]$, and by Proposition 3.2 and its dual that $[f_3 = g_3] \subsetneq [f = g]$. It remains to show that $[f_1 = g_1] \wedge [f_2 = g_2] \subsetneq [f_3 = g_3]$. Since $f_1\, \cancel{R}_{n+1}\, g_1$, it follows by Proposition 3.2 that $[f_1 = g_1] \subsetneq [f_3 = \bar{I}(g_3)\bar{F}(f_3)]$. Since $f_2\, \cancel{R}^*_{n+1}\, g_2$, it follows by Proposition 3.2*, that $[f_2 = g_2] \subsetneq [f_3 = \bar{I}(f_3)\bar{F}(g_3)]$. Hence $[f_1 = g_1] \wedge [f_2 = g_2] \subsetneq [f_3 = g_3]$.

Proposition 4.22. *Let* $|E(p)| = n \geq 3$, $p \sim_n q$, $p\, S_n\, q$, $p\, R_n{}^*\, q$, $|E(f)| = n + 1$, $f \sim_{n+1} g$, $f\, \cancel{R}_{n+1}\, g$. *Then* $[p = q] \not\subseteq [f = g]$.

Proof. By (3.7) it is enough to show that if $f = u\psi(p)v$ and $g = u\psi(q)v$, for some $u, v \in F'(X)$ and substitution ψ, then $f\ R_{n+1}\ g$.

There are two cases to consider. Assume first that $E(f) = E(u\psi(p))$. Then $\bar{I}(f)w = u\bar{I}(\psi(p))$, and $\bar{I}(g)w' = u\bar{I}(\psi(q))$ for some $w, w' \in F'(X)$. Since $p\, S_n\, q$ it follows that $\bar{I}(\psi(p)) \sim \bar{I}(\psi(q))$. Therefore from the definitions $\bar{I}(f) \sim \bar{I}(g)$, and hence in particular $f\ R_{n+1}\ g$.

Assume next that $E(f) \neq E(u\psi(p))$. In this case there exists a $v' \in F'(X)$ such that $f(0) = u\psi(p)v'$ and $g(0) = u\psi(q)v'$. Let φ be any primitive substitution in $(f(0) = g(0))$ by less than n variables. By Lemma 3.6 it follows that $\varphi(\psi(p)) \sim \varphi(\psi(q))$, and therefore that $\varphi(f(0)) \sim \varphi(g(0))$. From the definition of $\sim_n$ it then follows that $f(0) \sim_n g(0)$. Now it is trivial that $[p = q] \subsetneq [f(0) = g(0)]$, and therefore from the dual of Proposition 3.8 (i) it follows that $f(0)\, R_n{}^*\, g(0)$. Hence we can conclude from (2.20) that $f\ R_{n+1}\ g$.

Corollary 4.23. *Let* p_i, q_i, $(i = 1, 2, 3)$, *satisfy* $p_i \sim_n q_i$, $|E(p_i)| = n \geq 3$, $p_1\, S_n\, q_1$, $p_1\, \cancel{R}_n{}^*\, q_1$, $p_2\, \cancel{S}_n\, q_2$, $p_2\, S_n{}^*\, q_2$, $p_2\, R_n\, q_2$, $p_2\, R_n{}^*\, q_2$, $p_3\, S_n\, q_3$, $p_3\, R_n{}^*\, q_3$, $p_3\, \cancel{S}_n{}^*\, q_3$. *Then* $[p_1 = q_1] \vee [p_2 = q_2] \subseteq [p_3 = q_3]$ *and* $[p_1 = q_1] \vee [p_2 = q_2] \neq [p_3 = q_3]$.

Proof. That $[p_1 = q_1] \vee [p_2 = q_2] \subseteq [p_3 = q_3]$ is clear from the description of the n-skeleton. Let f, q satisfy $f \sim_{n+1} g$, $|E(f)| = n+1$, $f \mathcal{R}_{n+1} g$, $f \mathcal{R}^*_{n+1} g$. From Proposition 4.22, Proposition 4.21 (iii), and the dual of Proposition 4.21 (ii), it follows that $[p_1 = q_1] \vee [p_2 = q_2] \subseteq [f = g]$ and $[p_2 = q_2] \not\subseteq [f = g]$.

3. *Equational Classes Determined by One Equation*

PROPOSITION 4.24. *Let* $|E(p)| = n \geqslant 3$, $p \sim_n q$, $f \sim_n g$, $p \mathcal{R}_n{}^* q$ $f S_n g$. *Then* $[p = q] \subseteq [f = \bar{I}(g)\bar{F}(f)]$.

Note. This proposition is trivial if $|E(f)| = n$, for in that case $f S_n g$ states that $\bar{I}(f) \sim \bar{I}(g)$.

Proof. Since $f S_n g$, it follows from (2.19) that $f \sim_{n+1} \bar{I}(g)\bar{F}(f)$. Let p_1 and q_1 satisfy $|E(p_1)| = n+1$, $p_1 \sim_{n+1} q_1$, and $p_1 \mathcal{R}_{n+1} q_1$. Then from Proposition 4.1 and the dual of Proposition 4.21 (ii), it follows that $[p = q] \subseteq [p_1 = q_1] \subseteq [f = \bar{I}(g)\bar{F}(f)]$.

DEFINITION 4.25. Let $f \sim_n g$ and $n \geqslant 3$. Then

(i) $f T_n g$ iff $f R_{n+1} \bar{I}(g)\bar{F}(f)$

(ii) $f T_n{}^* g$ iff $f^* T_n g^*$.

If $f T_n g$, it follows that $f \sim_{n+1} \bar{I}(g)\bar{F}(f)$ but not necessarily that $f \sim_{n+1} g$.

$$\left.\begin{array}{llll} f\,T_n\,g & \text{implies} & f\,S_n\,g & (\text{for all } n \geqslant 3). \\ f\,T_n{}^*\,g & \text{implies} & f\,S_n{}^*\,g & (\text{for all } n \geqslant 3). \end{array}\right. \tag{4.26}$$

$$f\,T_n\,g \quad \text{and} \quad f\,T_n{}^*\,g \quad \text{imply} \quad f \sim_{n+1} g. \tag{4.27}$$

PROPOSITION 4.28. *For* $n \geqslant 3$, *each of the following conditions is sufficient for* $[p = q] \subseteq [f = \bar{I}(\bar{g})\bar{F}(f)]$

(i) $|E(p)| = n$, $p \sim_n q$, $p \not\sim_{n+1} q$, $f \sim_n g$, $f T_n g$.

(ii) $p \sim_n q$, $p \mathcal{T}_n q$, $p S_n q$, $|E(f)| = n+1$, $f \sim_{n+1} g$.

Proof. (i) Let p_1 and q_1 satisfy $|E(p_1)| = n+1$, $p_1 \mathcal{S}_{n+1} q_1$, $p_1 R_{n+1} q_1$, $p_1 S^*_{n+1} q_1$. Now $f R_{n+1} \bar{I}(g)\bar{F}(f)$, and $p_1 \mathcal{S}_{n+1} q_1$, and therefore by Proposition 4.20 it follows that $[p_1 = q_1] \subseteq [f = \bar{I}(g)\bar{F}(f)]$. Since $p \not\sim_{n+1} q$, and $|E(p)| = n$, either $p \mathcal{S}_n q$ or $p \mathcal{S}_n{}^* q$. Therefore, since $p_1 R_{n+1} q_1$ and $p_1 R^*_{n+1} q_1$, it follows from Proposition 4.21 (i) or its dual that $[p = q] \subseteq [p_1 = q_1]$. Hence $[p = q] \subseteq [f = \bar{I}(g)\bar{F}(f)]$.

(ii) Since $p S_n q$, it follows from (2.19) that $p \sim_{n+1} \bar{I}(q)\bar{F}(p)$. Moreover, since $p \mathcal{T}_n q$, it follows that $p \mathcal{R}_{n+1} \bar{I}(q)\bar{F}(p)$. Therefore by Proposition 4.1 we can conclude that $[p = \bar{I}(q)\bar{F}(p)] \subseteq [f = \bar{I}(g)\bar{F}(f)]$, and hence $[p = q] \subseteq [f = \bar{I}(g)\bar{F}(f)]$.

Proposition 4.29. *Let* f_i, g_i $(i = 0, 1)$, *satisfy* $|E(f_0)| = n \geqslant 3, f_i \sim_n g_i$. *Then each of the following conditions implies* $[f_0 = g_0] = [f_1 = g_1]$.

(1) $f_i \not{R}_n g_i, \quad f_i \not{R}_n{}^* g_i \quad (i = 0, 1)$

(2) $f_i R_n g_i, \quad f_i \not{S}_n g_i, \quad f_i \not{R}_n{}^* g_i \quad (i = 0, 1)$

(3) $f_i R_n{}^* g_i, \quad f_i \not{S}_n{}^* g_i, \quad f_i \not{R}_n g_i \quad (i = 0, 1)$

(4) $f_i S_n g_i, \quad f_i \not{R}_n{}^* g_i \quad (i = 0, 1)$

(5) $f_i S_n{}^* g_i, \quad f_i \not{R}_n g_i \quad (i = 0, 1)$

(6) $f_i \not{S}_n g_i, \quad f_i \not{S}_n{}^* g_i, \quad f_i R_n g_i, \quad f_i R_n{}^* g_i \quad (i = 0, 1)$

(7) $f_i R_n{}^* g_i, \quad f_i \not{S}_n{}^* g_i, \quad f_0 S_n g_0, \quad f_1 T_n g_1 \quad (i = 0, 1)$

(8) $f_i R_n g_i, \quad f_i \not{S}_n g_i, \quad f_0 S_n{}^* g_0, \quad f_1 T_n{}^* g_1 \quad (i = 0, 1)$

Moreover if p *and* q *satisfy* $|E(p)| = n + 1$, $p \sim_{n+1} q$, *then either of the following conditions implies* $[f_1 = g_1] = [f_0 = g_0] \wedge [p = q]$.

(9) $f_i R_n{}^* g_i, \ f_i \not{S}_n{}^* g_i, \ f_i S_n g_i, \quad f_1 \not{T}_n g_1, \quad p \not{R}_{n+1} q, \ p S_{n+1}^* q \quad (i = 0, 1)$

(10) $f_i R_n g_i, \quad f_i \not{S}_n g_i, \quad f_i S_n{}^* g_i, \ f_1 \not{T}_n{}^* g_1, \ p \not{R}_{n+1}^* q, \ p S_{n+1} q \quad (i = 0, 1)$

The ten equational classes determined by this proposition are mutually distinct.

Proof. The proof consists for the most part of listing those proportions already proved which can be applied.

(1) Propositions 4.1 and 4.1*

(2) Propositions 4.1* and 4.20

(3) dual of (2)

(4) Propositions 4.1* and 4.24

(5) dual of (4)

(6) Propositions 4.20 and 4.20*

(7) Propositions 4.28 (i) and 4.20*

(8) dual of (7)

(9) Since $f_0 S_n g_0$ and $p S_{n+1}^* q$ it follows from Propositions 4.20* and 4.28 (ii) that $[f_1 = g_1] \subseteq [f_0 = g_0] \wedge [p = q]$. Since $f_1 \sim_{n+1} \bar{I}(g_1) \bar{F}(f_1)$, it follows from Proposition 4.1 that $[f_0 = g_0] \subseteq [f_1 = \bar{I}(g_1) \bar{F}(f_1)]$. By Proposition 4.20*, $[p = q] \subseteq [g_1 = \bar{I}(g_1) \bar{F}(f_1)]$, and therefore $[f_0 = g_0] \wedge [p = q] \subseteq [f_1 = g_1]$.

(10) dual of (9).

The fact that the classes are mutually distinct follows from Theorem 3.9, and Corollary 4.23.

PROPOSITION 4.30 (Tamura [21, Lemma 13]). *Let* $f \sim_2 g$, *and let* a, b, $c \in X$.

(1) *If* $E(f) \neq E(g)$, $H(f) \neq H(g)$, $H^*(f) \neq H^*(g)$, *then* $[f = g] = [a = b]$.

(2) *If* $E(f) \neq E(g)$, $H(f) = H(g)$, $H^*(f) \neq H^*(g)$, *then* $[f = g] = [ab = a]$.

(3) *If* $E(f) \neq E(g)$, $H(f) \neq H(g)$, $H^*(g) = H^*(g)$, *then* $[f = g] = [ab = b]$.

(4) *If* $E(f) \neq E(g)$, $H(f) = H(g)$, $H^*(f) = H^*(g)$, *then* $[f = g] = [aba = a]$.

(5) *If* $E(f) = E(g)$, $H(f) \neq H(g)$, $H^*(f) \neq H^*(g)$, *then* $[f = g] = [ab = ba]$.

(6) *If* $E(f) = E(g)$, $H(f) = H(g)$, $H^*(f) \neq H^*(g)$, $I(f) = I(g)$, *then* $[f = g] = [aba = ab]$.

(7) *If* $E(f) = E(g)$, $H(f) \neq H(g)$, $H^*(f) = H^*(g)$, $F(f) = F(g)$, *then* $[f = g] = [aba = ba]$.

(8) *If* $E(f) = E(g)$, $H(f) = H(g)$, $H^*(f) \neq H^*(g)$, $I(f) \neq I(g)$, *then* $[f = g] = [abc = acb]$.

(9) *If* $E(f) = E(g)$, $H(f) \neq H(g)$, $H^*(f) = H^*(g)$, $F(f) \neq F(g)$, *then* $[f = g] = [abc = bac]$.

Moreover the nine equational classes determined by this proposition are mutually distinct.

We will not give a proof of this proposition since a proof is given by Tamura [21] in the proof of his Lemma 13 and the subsequent remarks. Moreover, a proof is fairly easy to reconstruct once the proposition has been stated. Notice that statement (14.10) of Tamura's Lemma 13 (i.e., if $E(f) = E(g)$, $H(f) = H(g)$, and $H^*(f) = H^*(g)$, then $[xyzx = xzyx] \subseteq [f = g]$, where x, y, $z \in X$) is a consequence of our Proposition 4.1 and its dual in the case $n = 3$.

In Propositions 4.29 and 4.30 we have given an exhaustive list of conditions which equations on idempotent semigroups may satisfy. The inclusions which exist among the equational classes given in Proposition 4.30 are easily determined, and the inclusions between these elements and the other classes determined by a single equation are given by the following proposition (and its dual).

PROPOSITION 4.31. *Let $a, b, c \in X$, and let $f_i \sim_3 g_i$, $|E(f_i)| = 3$, $i = 1, 2$, $f_1\, \mathcal{S}_3\, g_1$, $f_1\, R_3\, g_1$, $f_1\, \mathcal{R}_3{}^*\, g_1$, $f_2\, \mathcal{R}_3\, g_2$, $f_2\, \mathcal{R}_3{}^*\, g_2$. Then $[abc = acb] \subseteq [f_2 = g_2]$, $[aba = a] \subseteq [f_2 = g_2]$, $[aba = ab] \subseteq [f_1 = g_1]$, and $[aba = ab] \nsubseteq [f_2 = g_2]$.*

The proof of this result is also given in Tamura [21], but in any case a proof is fairly easy to reconstruct. Notice that for $(f_1 = g_1)$ we may take the equation $(abc = abac)$, and for $(f_2 = g_2)$ the equation $(abca = acba)$.

All of these results, that is all nontrivial inclusions among equational classes of idempotent semigroups, are summarized in Fig. 2.

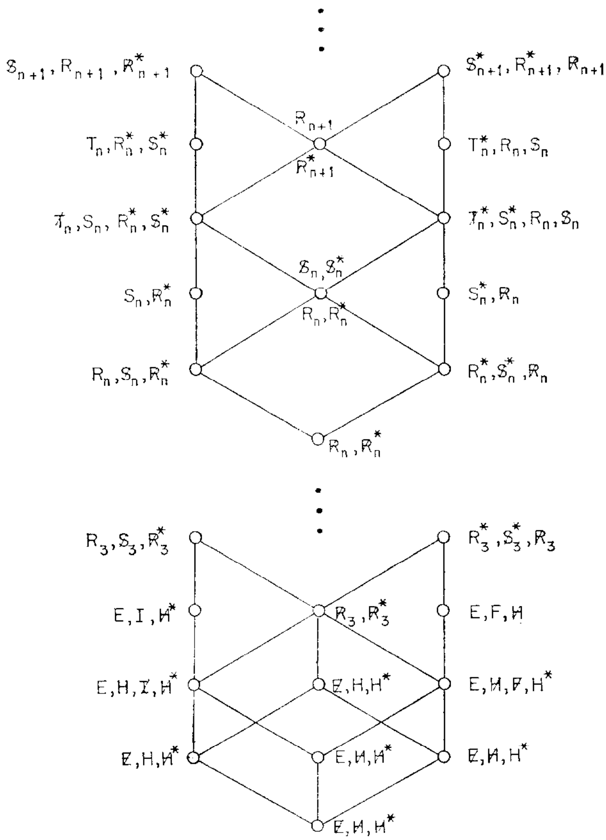

FIG. 2. The lattice.

4. *Equational Classes Determined by Arbitrarily Many Equations*

In order to show that the whole lattice has been described in Fig. 2, we now show that each equational class is determined by a single equation.

PROPOSITION 4.32. *Every equational class determined by finitely many equations is determined by one equation.*

Proof. Let $f_i \sim_3 g_i$ $(i = 1, 2)$. Then $E(f_i) = E(g_i)$, $H(f_i) = H(g_i)$, and $H^*(f_i) = H^*(g_i)$. Assume that $E(f_1) \cap E(f_2) = \varnothing$. It is trivial that $[f_1 = g_1] \wedge [f_2 = g_2] \subseteq [f_1 f_2 = g_1 g_2]$. Moreover, if φ_1 is a substitution which satisfies $\varphi_1(x) = H^*(f_1)$ for all $x \in E(f_2)$, $\varphi_1(y) = y$ for all $y \in E(f_1)$, then $\varphi_1(f_1 f_2) = f_1$ and $\varphi_1(g_1 g_2) = g_1$. Therefore $[f_1 f_2 = g_1 g_2] \subseteq [f_1 = g_1]$. Similarly, if φ_2 satisfies $\varphi_2(x) = H(f_2)$ for all $x \in E(f_1)$, $\varphi_2(y) = y$ for all $y \in E(f_2)$, then it follows that $[f_1 f_2 = g_1 g_2] \subseteq [f_2 = g_2]$.

If $f_1 \not\sim_3 g_1$, or $f_2 \not\sim_3 g_2$, then $[f_1 = g_1] \wedge [f_2 = g_2] = [p = q] \wedge [p_1 = q_1]$, where either both $[p = q]$ and $[p_1 = q_1]$ are equational classes given in Proposition 4.30, or $[p_1 = q_1]$ is one of these classes, and p and q satisfy $p \sim_3 q$ and one of the following:

(i) $p \not{S}_3 q$, $p R_3 q$, $p \not{R}_3{}^* q$

(ii) $p \not{S}_3{}^* q$, $p R_3{}^* q$, $p \not{R}_3 q$

(iii) $p \not{R}_3 q$, $p \not{R}_3{}^* q$.

It is straightforward to check that all such meets are classes determined by a single equation.

THEOREM 4.33. *Every equational class of idempotent semigroups is determined by one equation.*

Proof. The poset of equational classes which are determined by one equation is a lattice which satisfies the descending chain condition. Moreover, every set of incomparable elements in this lattice is finite. It follows that every meet in the lattice of equational classes is a finite meet in the lattice of classes determined by one equation, and therefore determined by one equation (by Proposition 4.32).

From this theorem it follows that the lattice of equational classes of idempotent semigroups has been completely described.

ACKNOWLEDGMENT

The author would like to thank his thesis supervisor Professor G. Bruns. He would also like to thank the National Research Council of Canada for its financial support.

REFERENCES

1. K. A. BAKER, Equational classes of modular lattices (prepublication copy).
2. G. BIRKHOFF, On the structure of abstract algebras, *Proc. Cambridge Phil. Soc.* **31** (1935), 433–454.
3. T. C. BROWN, On the finiteness of semigroups in which $x^r = x$, *Proc. Cambridge Phil. Soc.* **60** (1964), 1028–1029.
4. P. M. COHN, "Universal Algebra," Harper and Row, New York, 1965.
5. T. EVANS, The number of semigroup varieties, *Quart. J. Math. Oxford* (2) **19** (1968), 335.
6. J. A. GERHARD, The lattice of equational classes of idempotent semigroups, *Notices Amer. Math. Soc.* **15** (1968), 553.
7. G. GRÄTZER, "Universal Algebra," Van Nostrand, Princeton, N. J., 1968.
8. G. GRÄTZER, Equational classes of lattices, *Duke Math. J.* **33** (1966), 613–622.
9. J. A. GREEN AND D. REES, On semigroups in which $x^r = x$, *Proc. Cambridge Phil. Soc.* **48** (1952), 35–40.
10. E. JACOBS AND R. SCHWABAUER, The lattice of equational classes of algebras with one unary operation, *Amer. Math. Mon.* **71** (1964), 151–154.
11. B. JÓNSSON, Algebras whose congruence lattices are distributive, *Math. Scand.* **21** (1967), 110–121.
12. B. JÓNSSON, Equational classes of lattices (prepublication copy).
13. J. KALICKI, The number of equationally complete classes of equations, *Indag. Math.* **17** (1955), 660–662.
14. J. KALICKI AND D. SCOTT, Equational completeness of abstract algebras, *Indag. Math.* **17**(1955), 650–659.
15. N. KIMURA, Note on idempotent semigroups, IV, Identities of three variables, *Proc. Japan Acad.* **34** (1958), 121–123.
16. N. KIMURA, The structure of idempotent semigroups I, *Pac. J. Math.* **8** (1958), 257–275.
17. R. MCKENZIE, On equational theories of lattices (prepublication copy).
18. D. MCLEAN, Idempotent semigroups, *Amer. Math. Mon.* **61** (1954), 110–113.
19. E. NELSON, Equational classes of semigroups, *Notices Amer. Math. Soc.* **14** (1967), 543.
20. R. J. SCHWABAUER, The lattice of closed sets of commutative semigroup equations (prepublication copy).
21. T. TAMURA, Attainability of systems of identities of semigroups, *J. Algebra* **3** (1966), 261–276.
22. A. TARSKI, Contributions to the theory of models, *Indag. Math.* **16** (1954), 572–588, and **17** (1955), 56–64.